

# Some Notes on Finite Sets

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# 1 Introduction

For sets  $E$  and  $F$  we write  $E \approx F$  if there exists a bijective mapping  $f : E \rightarrow F$ . The standard definition of a set  $A$  being finite is that  $A \approx [n]$  for some  $n \in \mathbb{N}$ , where  $\mathbb{N} = \{0, 1, \dots\}$  is the set of natural numbers,  $[0] = \emptyset$ , and  $[n] = \{0, 1, \dots, n-1\}$  for each  $n \in \mathbb{N} \setminus \{0\}$ .

This seems straightforward enough until it is perhaps recalled how much effort is required to give a rigorous definition of the sets  $[m]$ ,  $m \in \mathbb{N}$ . Moreover, it is often not that easy to apply. For example, try giving, without too much thought, a proof of the fact that any injective mapping of a finite set into itself is bijective.

A more fundamental objection to the standard definition is its use of the infinite set  $\mathbb{N}$ , which might even give the impression that finite sets can only be defined with the help of such a set. This is certainly not the case and in what follows we are going to work with one of several possible definitions not involving the natural numbers. The definition introduced below is usually called *Kuratowski-finiteness* [4] and it is essentially that employed by Whitehead and Russell in *Principia Mathematica* [11]. We will also make use of a characterisation of finite sets due to Tarski [9]. There are similar approaches which also appeared in the first decades of the previous century and excellent treatments of this topic can be found, for example, in Levy [6] and Suppes [8].

These notes aim to give a gentle account of one approach to the theory of finite sets without making use of the natural numbers. They were written to be used as the basis for a student seminar. There are no real prerequisites except for a certain familiarity with the kind of mathematics seen in the first couple of years of a university mathematics course. In particular, only naive set theory will be encountered here and so it is not assumed that the reader has taken a course on axiomatic set theory. We assume that the reader has heard of the axiom of choice. This is not involved when dealing only with finite sets. Its use will be pointed out on the couple of occasions when a statement (involving infinite sets) depends on this axiom.

There is one important point which should be mentioned. We will be dealing with mappings defined on the collection of all finite sets and this collection is too large to be considered a set, meaning that treating it as a set might possibly lead to various paradoxes. Such a large collection is called a *class* and in particular the class of all finite sets will be denoted by  $\text{Fin}$ . However, as far as what is to be found in these notes, there is no problem in treating classes as if they were just sets.

It could be objected that, after having rejected the infinite set  $\mathbb{N}$  as a means of introducing finite sets, we now resort to objects which are so large that they

cannot even be considered to be sets. But mappings defined on classes can make sense without involving an infinite set. An important example is the mapping  $\sigma : \text{Fin} \rightarrow \text{Fin}$  defined by  $\sigma(A) = A \cup \{A\}$  for each finite set  $A$ . This mapping  $\sigma$  will form the basis for defining the finite ordinals.

The power set of a set  $E$ , i.e., the set of all its subsets, will be denoted by  $\mathcal{P}(E)$  and the set of non-empty subsets by  $\mathcal{P}_0(E)$ . If  $E$  is a set and  $\mathcal{S}$  is a subset of  $\mathcal{P}(E)$  then  $\mathcal{S}^p$  will be used to denote the set of subsets in  $\mathcal{S}$  which are proper subsets of  $E$ .

Finite sets will be defined here in terms of what is known as an inductive system, where a subset  $\mathcal{S}$  of  $\mathcal{P}(E)$  is called an *inductive  $E$ -system* if  $\emptyset \in \mathcal{S}$  and  $F \cup \{e\} \in \mathcal{S}$  for all  $F \in \mathcal{S}^p$ ,  $e \in E \setminus F$ . In particular,  $\mathcal{P}(E)$  is itself an inductive  $E$ -system.

The definition of being finite which will be used here is the following; [4], [11]:

A set  $E$  is defined to be *finite* if  $\mathcal{P}(E)$  is the only inductive  $E$ -system.

In Theorem 1.1 we show that the above definition of being finite is equivalent to the standard definition given in terms of the natural numbers.

**Lemma 1.1** *The empty set  $\emptyset$  is finite. Moreover, for each finite set  $A$  and each element  $a \notin A$  the set  $A \cup \{a\}$  is finite.*

*Proof* The empty set  $\emptyset$  is finite since  $\mathcal{P}(\emptyset) = \{\emptyset\}$  is the only non-empty subset of  $\mathcal{P}(\emptyset)$ . Now consider a finite set  $A$  and  $a \notin A$ . Put  $B = A \cup \{a\}$  and let  $\mathcal{R}$  be an inductive  $B$ -system. Then  $\mathcal{S} = \mathcal{R} \cap \mathcal{P}(A)$  is clearly an inductive  $A$ -system and thus  $\mathcal{S} = \mathcal{P}(A)$  (since  $A$  is finite), i.e.,  $\mathcal{P}(A) \subset \mathcal{R}$ . Moreover,  $A' \cup \{a\} \in \mathcal{R}$  for all  $A' \in \mathcal{P}(A)$ , since  $\mathcal{R}$  is an inductive  $B$ -system and  $\mathcal{P}(A) \subset \mathcal{R}$ . This implies that  $\mathcal{R} = \mathcal{P}(B)$  and hence that  $B = A \cup \{a\}$  is finite.  $\square$

Note that an arbitrary intersection of inductive  $E$ -systems is again an inductive  $E$ -system and so there is a least inductive  $E$ -system (namely the intersection of all such inductive  $E$ -systems). Thus if the least inductive  $E$ -system is denoted by  $\mathcal{I}_0(E)$  then a set  $E$  is finite if and only if  $\mathcal{I}_0(E) = \mathcal{P}(E)$ .

The set of finite subsets of a set  $E$  will be denoted by  $\text{Fin}(E)$ .

**Lemma 1.2** *For each set  $E$  the least inductive  $E$ -system is exactly the set of finite subsets of  $E$ , i.e.,  $\mathcal{I}_0(E) = \text{Fin}(E)$ . In particular, a set  $E$  is finite if and only if every inductive  $E$ -system contains  $E$ .*

*Proof* By Lemma 1.1  $\text{Fin}(E)$  is an inductive  $E$ -system and hence  $\mathcal{I}_0(E) \subset \text{Fin}(E)$ . Conversely, if  $A \in \text{Fin}(E)$  then  $A \in \mathcal{P}(A) = \mathcal{I}_0(A) \subset \mathcal{I}_0(E)$  and therefore also  $\text{Fin}(E) \subset \mathcal{I}_0(E)$ . In particular, it follows that if every inductive  $E$ -system contains  $E$  then  $E \in \mathcal{I}_0(E) = \text{Fin}(E)$ , and hence  $E$  is finite. Clearly if  $E$  is finite then  $E \in \mathcal{P}(E) = \mathcal{I}_0(E)$  and so every inductive  $E$ -system contains  $E$ .  $\square$

**Proposition 1.1** *Every subset of a finite set  $A$  is finite.*

*Proof* By Lemma 1.2  $\text{Fin}(A) = \mathcal{P}(A)$ , and hence every subset of  $A$  is finite.  $\square$

We next show that the definition of being finite employed here is equivalent to the standard definition. To help distinguish between these two definitions let us call sets which are finite according to the standard definition  $\mathbb{N}$ -finite. Thus a set  $A$  is  $\mathbb{N}$ -finite if and only if there exists a bijective mapping  $h : [n] \rightarrow A$  for some  $n \in \mathbb{N}$ . (When working with this definition we assume the reader is familiar with the properties of the sets  $[n]$ ,  $n \in \mathbb{N}$ .)

**Theorem 1.1** *A set is finite if and only if it is  $\mathbb{N}$ -finite.*

*Proof* Let  $A$  be a finite set and let  $\mathcal{S} = \{B \in \mathcal{P}(A) : B \text{ is } \mathbb{N}\text{-finite}\}$ . Clearly  $\emptyset \in \mathcal{S}$ , so consider  $B \in \mathcal{S}^p$ , let  $a \in A \setminus B$  and put  $B' = B \cup \{a\}$ . By assumption there exists  $n \in \mathbb{N}$  and a bijective mapping  $h : [n] \rightarrow B$  and the mapping  $h$  can be extended to a bijective mapping  $h' : [n+1] \rightarrow B'$  by putting  $h'(n) = a$ ; hence  $B' \in \mathcal{S}$ . It follows that  $\mathcal{S}$  is an inductive  $A$ -system and hence  $\mathcal{S} = \mathcal{P}(A)$ , since  $A$  is finite. In particular  $A \in \mathcal{S}$ , i.e.,  $A$  is  $\mathbb{N}$ -finite. This shows that each finite set is  $\mathbb{N}$ -finite.

Now let  $A$  be  $\mathbb{N}$ -finite, so there exists  $n \in \mathbb{N}$  and a bijective mapping  $h : [n] \rightarrow A$ . Let  $\mathcal{S}$  be an inductive  $A$ -system. For each  $k \in [n+1] = [n] \cup \{n\}$  put  $A_k = h([k])$ . Then  $A_0 = h(\emptyset) = \emptyset$ ,  $A_n = h([n]) = A$  and for each  $k \in [n]$

$$A_{k+1} = h([k+1]) = h([k]) \cup h(\{k\}) = A_k \cup \{a_k\},$$

where  $a_k = h(k)$ . Thus  $A_0 = \emptyset \in \mathcal{S}$  and if  $A_k \in \mathcal{S}$  for some  $k \in [n]$  then also  $A_{k+1} = A_k \cup \{a_k\} \in \mathcal{S}$ , since  $\mathcal{S}$  is an inductive  $A$ -system. This means that if we put  $J = \{k \in [n+1] : A_k \in \mathcal{S}\}$  then  $0 \in J$  and  $k+1 \in J$  whenever  $k \in J$  for some  $k \in [n]$ . It follows that  $J = [n+1]$  (insert your own proof of this fact here) and in particular  $n \in J$ , i.e.,  $A = A_n \in \mathcal{S}$ . This shows that every inductive  $A$ -system contains  $A$  and therefore by Lemma 1.2  $A$  is finite.  $\square$

There is a further characterisation of finite sets due to Tarski [9] which will be very useful for establishing properties of finite sets. If  $\mathcal{C}$  is a non-empty subset of  $\mathcal{P}(E)$  then  $C \in \mathcal{C}$  is said to be *minimal* if  $D \notin \mathcal{C}$  for each proper subset  $D$  of  $C$ .

**Proposition 1.2** *A set  $E$  is finite if and only if each non-empty subset of  $\mathcal{P}(E)$  contains a minimal element.*

*Proof* Let  $A$  be a finite set and let  $\mathcal{S}$  be the set consisting of those elements  $B \in \mathcal{P}(A)$  such that each non-empty subset of  $\mathcal{P}(B)$  contains a minimal element. Then  $\emptyset \in \mathcal{S}$ , since the only non-empty subset of  $\mathcal{P}(\emptyset)$  is  $\{\emptyset\}$  and then  $\emptyset$  is the required minimal element. Let  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ , and  $\mathcal{C}$  be a non-empty subset of  $\mathcal{P}(B \cup \{a\})$ . Put  $\mathcal{D} = \mathcal{C} \cap \mathcal{P}(B)$ ; there are two cases:

( $\alpha$ )  $\mathcal{D} \neq \emptyset$ . Here  $\mathcal{D}$  is a non-empty subset of  $\mathcal{P}(B)$  and thus contains a minimal element  $D$  which is then a minimal element of  $\mathcal{C}$ , since each set in  $\mathcal{C} \setminus \mathcal{D}$  contains  $a$  and so is not a proper subset of  $D$ .

( $\beta$ )  $\mathcal{D} = \emptyset$  (and so each set in  $\mathcal{C}$  contains  $a$ ). Put  $\mathcal{F} = \{C \subset B : C \cup \{a\} \in \mathcal{C}\}$ ; then  $\mathcal{F}$  is a non-empty subset of  $\mathcal{P}(B)$  and thus contains a minimal element  $F$ . It follows that  $F' = F \cup \{a\}$  is a minimal element of  $\mathcal{C}$ : A proper subset of  $F'$  has either the form  $C$  with  $C \subset F$ , in which case  $C \notin \mathcal{C}$  (since each set in  $\mathcal{C}$  contains  $a$ ), or has the form  $C \cup \{a\}$  with  $C$  a proper subset of  $F$  and here  $C \cup \{a\} \notin \mathcal{C}$ , since  $C \notin \mathcal{F}$ .

This shows that  $B \cup \{a\} \in \mathcal{S}$  and thus that  $\mathcal{S}$  is an inductive  $A$ -system. Hence  $\mathcal{S} = \mathcal{P}(A)$  and in particular  $A \in \mathcal{S}$ , i.e., each non-empty subset of  $\mathcal{P}(E)$  contains a minimal element.

Conversely, suppose  $E$  is not finite and let  $\mathcal{C} = \{C \in \mathcal{P}(E) : C \text{ is not finite}\}$ ; then  $\mathcal{C}$  is non-empty since it contains  $E$ . However  $\mathcal{C}$  cannot contain a minimal element: If  $D$  were a minimal element of  $\mathcal{C}$  then  $D \neq \emptyset$ , since  $\emptyset$  is finite. Choose  $d \in D$ ; then  $D \setminus \{d\}$  is a proper subset of  $D$  and thus  $D \setminus \{d\} \notin \mathcal{C}$ , i.e.,  $D \setminus \{d\}$  is finite. But then by Lemma 1.1  $D = (D \setminus \{d\}) \cup \{d\}$  would be finite.  $\square$

If  $\mathcal{C}$  is a non-empty subset of  $\mathcal{P}(E)$  then  $C \in \mathcal{C}$  is said to be *maximal* if  $C' = C$  whenever  $C' \in \mathcal{C}$  with  $C \subset C' \subset E$ .

**Proposition 1.3** *If  $A$  is finite then every non-empty subset  $\mathcal{C}$  of  $\mathcal{P}(A)$  contains a maximal element.*

*Proof* The set  $\mathcal{D} = \{A \setminus C : C \in \mathcal{C}\}$  is also a non-empty subset of  $\mathcal{P}(A)$  and therefore by Proposition 1.2 it contains a minimal element which has the form  $A \setminus C$  with  $C \in \mathcal{C}$ . Hence  $\{D \in \mathcal{C} : C \subset D \subset A\} = \{C\}$  and so  $C$  is maximal.  $\square$

We end the Introduction by outlining some of the main results to be found in these notes.

In Section 2 we establish the basic properties of finite sets. Most of these simply confirm that finite sets are closed under the usual set-theoretic operations. More precisely, if  $A$  and  $B$  are finite sets then their union  $A \cup B$ , their product  $A \times B$  and  $B^A$  (the set of all mappings from  $A$  to  $B$ ) are all finite sets. Moreover, the power set  $\mathcal{P}(A)$  is finite and (Proposition 1.1) any subset of a finite set is finite.

Apart from these closure properties there are two properties which depend crucially on the set involved being finite. The first is given in Theorem 2.1 which states that if  $A$  is a finite set and  $f : A \rightarrow A$  is a mapping then  $f$  is injective if and only if it is surjective (and thus if and only if it is bijective).

Theorem 2.1 implies the set  $\mathbb{N}$  of natural numbers is infinite (i.e., it is not finite), since the successor mapping  $\mathbf{s} : \mathbb{N} \rightarrow \mathbb{N}$  with  $\mathbf{s}(n) = n + 1$  for all  $n \in \mathbb{N}$  is injective but not surjective.

If  $E, F$  are any sets then we write  $E \approx F$  if there exists a bijective mapping  $f : E \rightarrow F$ . A direct corollary of Theorem 2.1 (Theorem 2.2) is that if  $B$  is a subset of a finite set  $A$  with  $B \approx A$  then  $B = A$ .

The second important property involving finite sets is Theorem 2.3, which states that if  $A$  and  $E$  are sets with  $A$  finite and if there exists either an injective mapping  $f : E \rightarrow A$  or a surjective mapping  $f : A \rightarrow E$  then  $E$  is finite.

Recall that  $\mathbf{Fin}$  denotes the class of all finite sets (which is itself too large to be considered a set). In Section 3 we introduce what will be called an assignment of finite sets in a triple  $\mathbb{I} = (X, f, x_0)$ , where  $X$  is some class of objects,  $f : X \rightarrow X$  is a mapping of the class  $X$  into itself and  $x_0$  is an object of  $X$ . Such a triple will be called an *iterator*. The results in this section will be applied in Section 4 to define the finite ordinals. In this case  $(X, f, x_0) = (\mathbf{Fin}, \sigma, \emptyset)$ , where  $\sigma : \mathbf{Fin} \rightarrow \mathbf{Fin}$  is the mapping given by  $\sigma(A) = A \cup \{A\}$  for each finite set  $A$ .

The archetypal example of an iterator whose first component is a set is  $(\mathbb{N}, \mathbf{s}, 0)$ , where the successor mapping  $\mathbf{s} : \mathbb{N} \rightarrow \mathbb{N}$  is given by  $\mathbf{s}(n) = n + 1$  for each  $n \in \mathbb{N}$ . However, we will also be dealing with examples in which the first component is a finite set. Let  $\mathbb{I} = (X, f, x_0)$  be an iterator. A mapping  $\omega : \mathbf{Fin} \rightarrow X$  will be called an *assignment of finite sets in  $\mathbb{I}$*  if  $\omega(\emptyset) = x_0$  and  $\omega(A \cup \{a\}) = f(\omega(A))$  for each finite set  $A$  and each element  $a \notin A$ .

For each finite set  $A$  denote the cardinality of  $A$  by  $|A|$  (with  $|A|$  defined as usual in terms of  $\mathbb{N}$ ). Then it is clear that the mapping  $|\cdot| : \mathbf{Fin} \rightarrow \mathbb{N}$  defines an assignment of finite sets in  $(\mathbb{N}, \mathbf{s}, 0)$  and that this is the unique such assignment.

Theorem 3.1 states that for each iterator  $\mathbb{I}$  there exists a unique assignment  $\omega$  of finite sets in  $\mathbb{I}$  and that if  $A$  and  $B$  are finite sets with  $A \approx B$  then  $\omega(A) = \omega(B)$ .

Let  $X_0 = \{x \in X : x = \omega(A) \text{ for some finite set } A\}$ . A subclass  $Y$  of  $X$  is said to be *f-invariant* if  $f(y) \in Y$  for all  $y \in Y$ . Lemma 3.4 shows that  $X_0$  is the least *f*-invariant subclass of  $X$  containing  $x_0$ .

The iterator  $\mathbb{I}$  is said to be *minimal* if the only *f*-invariant subclass of  $X$  containing  $x_0$  is  $X$  itself, thus  $\mathbb{I}$  is minimal if and only if  $X_0 = X$ . In particular, it is easy to see that the Principle of Mathematical Induction is exactly the requirement that the iterator  $(\mathbb{N}, s, 0)$  be minimal. If  $\mathbb{I}$  is minimal then Lemma 3.4 implies that for each  $x \in X$  there exists a finite set  $A$  with  $x = \omega(A)$ .

The iterator  $\mathbb{I}$  will be called *proper* if  $B_1 \approx B_2$  whenever  $B_1$  and  $B_2$  are finite sets with  $\omega(B_1) = \omega(B_2)$ . If  $\mathbb{I}$  is proper then by Theorem 3.1  $B_1 \approx B_2$  holds if and only if  $\omega(B_1) = \omega(B_2)$ .

$\mathbb{I}$  will be called a *Peano iterator* if it is minimal and  $\mathbb{N}$ -like, where  *$\mathbb{N}$ -like* means that the mapping  $f$  is injective and  $x_0 \notin f(X)$ . The Peano axioms thus require  $(\mathbb{N}, s, 0)$  to be a Peano iterator.

Theorem 3.2 states that a minimal iterator is proper if and only if it is a Peano iterator. A corollary is Theorem 3.3 (the recursion theorem for Peano iterators). It states that if  $\mathbb{I}$  is a Peano iterator then for each iterator  $\mathbb{J} = (Y, g, y_0)$  there exists a unique mapping  $\pi : X \rightarrow Y$  with  $\pi(x_0) = y_0$  such that  $\pi \circ f = g \circ \pi$ . The Peano axioms require  $(\mathbb{N}, s, 0)$  to be a Peano iterator and hence for each iterator  $\mathbb{J} = (Y, g, y_0)$  there exists a unique mapping  $\pi : \mathbb{N} \rightarrow Y$  with  $\pi(0) = y_0$  such that  $\pi \circ s = g \circ \pi$ . Theorem 3.4 shows that if  $\mathbb{I}$  is minimal then the class  $X$  is a finite set if and only if  $\mathbb{I}$  is not proper.

Theorem 3.6 is a result which guarantees the existence of mappings 'defined by recursion'. It states that if  $\mathbb{I} = (X, f, x_0)$  is a Peano iterator,  $Y$  and  $Z$  are classes and  $\beta : X \times Y \rightarrow Z$  and  $\alpha : X \times Y \times Z \rightarrow Z$  are mappings then there is a unique mapping  $\pi : X \times Y \rightarrow Z$  with  $\pi(x_0, y) = \beta(y)$  for all  $y \in Y$  such that  $\pi(f(x), y) = \alpha(x, y, \pi(x, y))$  for all  $x \in X, y \in Y$ .

Let  $\mathbb{I} = (X, f, x_0)$  and  $\mathbb{J} = (Y, g, y_0)$  be iterators; a mapping  $\mu : X \rightarrow Y$  is called a *morphism* from  $\mathbb{I}$  to  $\mathbb{J}$  if  $\mu(x_0) = y_0$  and  $g \circ \mu = \mu \circ f$ . This will also be expressed by saying that  $\mu : \mathbb{I} \rightarrow \mathbb{J}$  is a morphism. The iterators  $\mathbb{I}$  and  $\mathbb{J}$  are said to be *isomorphic* if there exists a morphism  $\mu : \mathbb{I} \rightarrow \mathbb{J}$  and a morphism  $\nu : \mathbb{J} \rightarrow \mathbb{I}$  such that  $\nu \circ \mu = \text{id}_X$  and  $\mu \circ \nu = \text{id}_Y$ . In particular, the mappings  $\mu$  and  $\nu$  are then both bijections. The iterator  $\mathbb{I}$  is said to be *initial* if for each iterator  $\mathbb{J}$  there is a unique morphism from  $\mathbb{I}$  to  $\mathbb{J}$ . Theorem 3.3 (the recursion theorem) thus states that a Peano iterator is initial. Let  $\mathbb{I}$  be initial and  $\pi : \mathbb{I} \rightarrow \mathbb{J}$  be the unique morphism. If  $\mathbb{J}$  is initial then  $\pi$  is an isomorphism and so  $\mathbb{I}$  and  $\mathbb{J}$  are isomorphic. Conversely, if  $\pi$  is an isomorphism then  $\mathbb{J}$  is initial. This shows that, up to isomorphism, there is a unique initial iterator. Of course, this is only true if an initial iterator exists, and



$(\mathbb{N}, \mathbf{s}, 0)$  is an initial iterator. In fact, in Section 4 we will exhibit another initial iterator  $\mathbb{O}_0 = (O, \sigma_0, \emptyset)$  which is defined only in terms of finite sets and makes no use of the infinite set  $\mathbb{N}$ . The elements of  $O$  are the finite ordinals.

Theorem 3.7 is a result of Lawvere [5] which shows the converse of the recursion theorem holds. That is, an initial iterator  $\mathbb{I} = (X, f, x_0)$  is a Peano iterator.

In Section 4 we study finite ordinals using the standard approach introduced by von Neumann [10]. Let  $\sigma : \mathbf{Fin} \rightarrow \mathbf{Fin}$  be the mapping given by  $\sigma(A) = A \cup \{A\}$  for each finite set  $A$ . If we iterate the operation  $\sigma$  starting with the empty set and label the resulting sets using the natural numbers then we obtain the following:

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= \sigma(0) = 0 \cup \{0\} = \emptyset \cup \{0\} = \{0\}, \\ 2 &= \sigma(1) = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\}, \\ 3 &= \sigma(2) = 2 \cup \{2\} = \{0, 1\} \cup \{2\} = \{0, 1, 2\}, \\ 4 &= \sigma(3) = 3 \cup \{3\} = \{0, 1, 2\} \cup \{3\} = \{0, 1, 2, 3\}, \\ 5 &= \sigma(4) = 4 \cup \{4\} = \{0, 1, 2, 3\} \cup \{4\} = \{0, 1, 2, 3, 4\}, \\ n + 1 &= \sigma(n) = n \cup \{n\} = \{0, 1, 2, \dots, n - 1\} \cup \{n\} = \{0, 1, 2, \dots, n\}. \end{aligned}$$

Denote by  $\mathbb{O}$  the iterator  $(\mathbf{Fin}, \sigma, \emptyset)$ . Then by Theorem 3.1 there exists a unique assignment  $\varrho$  of finite sets in  $\mathbb{O}$ . Thus  $\varrho : \mathbf{Fin} \rightarrow \mathbf{Fin}$  is the unique mapping with  $\varrho(\emptyset) = \emptyset$  and such that  $\varrho(A \cup \{a\}) = \sigma(\varrho(A))$  for each finite set  $A$  and each element  $a \notin A$ . Moreover, if  $A$  and  $B$  are finite sets with  $A \approx B$  then  $\varrho(A) = \varrho(B)$ . Theorem 4.1 states that  $\varrho(A) \approx A$  for each finite set and thus  $\varrho(A) = \varrho(B)$  if and only if  $A \approx B$ . Let  $O = \{B \in \mathbf{Fin} : B = \varrho(A) \text{ for some finite set } A\}$ . The elements of  $O$  will be called *finite ordinals*. Thus for each finite set  $A$  there exists a unique  $o \in O$  with  $o = \varrho(A)$ . By Lemma 3.4  $O$  is the least  $\sigma$ -invariant subclass of  $\mathbf{Fin}$  containing  $\emptyset$ . Let  $\sigma_0 : O \rightarrow O$  be the restriction of  $\sigma$  to  $O$ . Then  $\mathbb{O}_0 = (O, \sigma_0, \emptyset)$  is a minimal iterator.

In fact, Theorem 4.2 shows that  $\mathbb{O}_0$  is a Peano iterator. Thus by the recursion theorem it follows that for each iterator  $\mathbb{J} = (Y, g, y_0)$  there exists a unique mapping  $\pi : X \rightarrow Y$  with  $\pi(\emptyset) = y_0$  such that  $\pi \circ \sigma = g \circ \pi$ .

Proposition 4.1 states that for each finite set  $A$

$$\varrho(A) = \{o \in O : o = \varrho(A') \text{ for some proper subset } A' \text{ of } A\}.$$

It follows that if  $A$  is a finite set and  $B \subset A$ ; then  $\varrho(B) \subset \varrho(A)$ . It also follows that for each  $o \in O$

$$o = \{o' \in O : o' \text{ is a proper subset of } o\},$$

$$\sigma(o) = \{o' \in O : o' \text{ is a subset of } o\}.$$

Proposition 4.4 states that if  $o, o' \in O$  with  $o \neq o'$ , then either  $o$  is a proper subset of  $o'$  or  $o'$  is a proper subset of  $o$ .

We also look at two further Peano iterators  $\mathbb{U}_0$  and  $\mathbb{V}_0$ . They have nothing to do with ordinals, except that the iterator  $\mathbb{O}_0$  is involved in their definition. What they have in common with the iterator  $\mathbb{O}_0$  is that they are defined 'absolutely' and are in fact constructed solely from operations performed on the empty set  $\emptyset$ . The first of these iterators is the minimal iterator obtained from the iterator  $\mathbb{V} = (\text{Fin}, \beta, \emptyset)$ , where the mapping  $\beta : \text{Fin} \rightarrow \text{Fin}$  is given by  $\beta(A) = \{A\}$  for each finite set  $A$ .

Except for being a Peano iterator the iterator  $\mathbb{V}_0$  has none of the properties enjoyed by  $\mathbb{O}_0$ . It corresponds to the perhaps most primitive method of counting by representing the number  $n$  with something like  $n$  marks, in this case the empty set enclosed in  $n$  braces.

We can improve the situation somewhat by applying Theorem 3.5 to the iterator  $\mathbb{V}_0$ . This results in the iterator  $\mathbb{U} = (\text{Fin}, \alpha, \emptyset)$ , where the mapping  $\alpha : \text{Fin} \rightarrow \text{Fin}$  is given by  $\alpha(A) = \{\emptyset\} \cup \beta(A)$  for each finite set  $A$ . The corresponding minimal iterator  $\mathbb{U}_0 = (U, \alpha_0, \emptyset)$  is a Peano iterator and it has the property that if  $u, v \in U$  with  $u \neq v$  then either  $u$  is a proper subset of  $v$  or  $v$  is a proper subset of  $u$ .

Theorems 3.2 and 3.4 imply that for a minimal iterator  $\mathbb{I} = (X, f, x_0)$  there are two mutually exclusive possibilities: Either  $\mathbb{I}$  is a Peano iterator or  $X$  is a finite set. In Section 5 we deal with case when  $X$  is a finite set.

Let  $\mathbb{I} = (X, f, x_0)$  be a minimal iterator with  $X$  a finite set. For each  $x \in X$  let  $X_x$  be the least  $f$ -invariant subset of  $X$  containing  $x$  and let  $f_x : X_x \rightarrow X_x$  be the restriction of  $f$  to  $X_x$ . Thus  $\mathbb{I}_x = (X_x, f_x, x)$  is a minimal iterator. An element  $x \in X$  is said to be *periodic* if  $x \in f_x(X_x)$  and so by Proposition 3.3 and Theorem 2.1  $x$  is periodic if and only if  $f_x$  is a bijection. Theorem 5.1 states that:

- (1) Let  $X_P = \{x \in X : x \text{ is periodic}\}$ . Then  $X_P$  is non-empty and  $X_x = X_y$  for all  $x, y \in X_P$ . Thus  $f$  maps  $X_P$  bijectively onto itself.
- (2) Let  $X_N = \{x \in X : x \text{ is not periodic}\}$  and suppose  $X_N \neq \emptyset$ . Then  $f$  is injective on  $X_N$  and there exists a unique element  $u \in X_N$  such that  $f(u)$  is periodic. Moreover, there exists a unique element  $v \in X_P$  such that  $f(v) = f(u)$  and  $u$  and  $v$  are the unique elements of  $X$  with  $u \neq v$  such that  $f(u) = f(v)$ .

Statement (1) corresponds to the elementary fact that a mapping  $f : X \rightarrow X$  with  $X$  a finite set is eventually periodic.

In Section 6 we show how an addition and a multiplication can be defined for any minimal iterator. These operations are associative and commutative and can be specified by the rules (a0), (a1), (m0) and (m1) below, which are usually employed when defining the operations on  $\mathbb{N}$  via the Peano axioms.

Let  $\mathbb{I} = (X, f, x_0)$  be a minimal iterator with  $\omega$  the assignment of finite sets in  $\mathbb{I}$ . Theorem 6.1 states that there exists a unique binary operation  $\oplus$  on  $X$  such that

$$\omega(A) \oplus \omega(B) = \omega(A \cup B)$$

whenever  $A$  and  $B$  are disjoint finite sets. This operation  $\oplus$  is both associative and commutative,  $x \oplus x_0 = x$  for all  $x \in X$  and for all  $x_1, x_2 \in X$  there is a  $x \in X$  such that either  $x_1 = x_2 \oplus x$  or  $x_2 = x_1 \oplus x$ . Moreover,  $\oplus$  is the unique binary operation  $\oplus$  on  $X$  such that

$$(a0) \quad x \oplus x_0 = x \text{ for all } x \in X.$$

$$(a1) \quad x \oplus f(x') = f(x \oplus x') \text{ for all } x, x' \in X.$$

If  $f$  is injective then the cancellation law holds for  $\oplus$  (meaning that  $x_1 = x_2$  whenever  $x_1 \oplus x = x_2 \oplus x$  for some  $x \in X$ ).

If  $x_0 \in f(X)$  then by Theorem 3.4 and Proposition 3.4  $X$  is finite and  $f$  is bijective and here  $(X, \oplus, x_0)$  is the cyclic group generated by the element  $f(x_0)$ .

Theorem 6.2 states that there exists a unique binary operation  $\otimes$  on  $X$  such that

$$\omega(A) \otimes \omega(B) = \omega(A \times B)$$

for all finite sets  $A$  and  $B$ . This operation  $\otimes$  is both associative and commutative,  $x \otimes x_0 = x_0$  for all  $x \in X$  and  $x \otimes f(x_0) = x$  for all  $x \in X$  with  $x \neq x_0$  (and so  $f(x_0)$  is a multiplicative identity) and the distributive law holds for  $\oplus$  and  $\otimes$ :

$$x \otimes (x_1 \oplus x_2) = (x \otimes x_1) \oplus (x \otimes x_2)$$

for all  $x, x_1, x_2 \in X$ . Moreover,  $\otimes$  is the unique binary operation on  $X$  such that

$$(m0) \quad x \otimes x_0 = x_0 \text{ for all } x \in X.$$

$$(m1) \quad x \otimes f(x') = x \oplus (x \otimes x') \text{ for all } x, x' \in X.$$

We also look at the operation of exponentiation. Here we have to be more careful: For example,  $2 \cdot 2 \cdot 2 = 2$  in  $\mathbb{Z}_3$  and so  $2^3$  is not well-defined if the exponent 3 is considered as an element of  $\mathbb{Z}_3$  (since we would also have to have  $2^0 = 1$ ). However,

$2^3$  does make sense if 2 is considered as an element of  $\mathbb{Z}_3$  and the exponent 3 as an element of  $\mathbb{N}$ .

In general it is the case that if  $\mathbb{J} = (Y, g, y_0)$  is a Peano iterator then we can define an element of  $X$  which is ‘ $x$  to the power of  $y$ ’ for each  $x \in X$  and each  $y \in Y$  and this operation has the properties which might be expected.

Let  $\mathbb{J} = (Y, g, y_0)$  be a Peano iterator with  $\omega'$  the assignment of finite sets in  $\mathbb{J}$ . (As before  $\mathbb{I} = (X, f, , x_0)$  is assumed to be minimal with  $\omega$  the assignment of finite sets in  $\mathbb{I}$ .) Also let  $\oplus$  and  $\otimes$  be the operations given in Theorems 6.1 and 6.2 for the iterator  $\mathbb{I}$ .

Theorem 6.3 states that there exists a unique operation  $\uparrow : X \times Y \rightarrow X$  such that

$$\omega(A) \uparrow \omega'(B) = \omega(A^B)$$

for all finite sets  $A$  and  $B$ . This operation  $\uparrow$  satisfies

$$x \uparrow (y_1 \oplus y_2) = (x \uparrow y_1) \otimes (x \uparrow y_2)$$

for all  $x \in X$  and all  $y_1, y_2 \in Y$  and

$$(x_1 \otimes x_2) \uparrow y = (x_1 \uparrow y) \otimes (x_2 \uparrow y)$$

for all  $x_1, x_2 \in X$  and  $y \in Y$ . Moreover,  $\uparrow$  is the unique operation such that

(e0)  $x \uparrow y_0 = f(x_0)$  for all  $x \in X$ .

(e1)  $x \uparrow g(y) = x \otimes (x \uparrow y)$  for all  $x \in X, y \in Y$ .

In Section 7 we give alternative proofs for Theorem 6.1 and Theorem 6.2.

For a given finite set  $A$  Section 8 looks at the group  $S_A$  of bijective mappings  $\sigma : A \rightarrow A$  (with functional composition  $\circ$  as group operation and  $\text{id}_A$  as identity element. The elements of  $S_A$  are called *permutations*.

An element  $\tau$  of  $S_A$  is a *transposition* if there exist  $b, c \in A$  with  $b \neq c$  such that

$$\tau(x) = \begin{cases} c & \text{if } x = b, \\ b & \text{if } x = c, \\ x & \text{otherwise.} \end{cases}$$

Denote by  $F_2$  the multiplicative group  $\{+, -\}$  with  $+\cdot+ = -\cdot- = +$  and  $-\cdot+ = +\cdot- = -$ . For each element  $s \in F_2$  the other element will be denoted by

$-s$ . A mapping  $\sigma : S_A \rightarrow F_2$  is a *signature* if  $\sigma(\text{id}_A) = +$  and  $\sigma(\tau \circ f) = -\sigma(f)$  for each  $f \in S_A$  and for each transposition  $\tau$ .

Theorem 8.1 gives a proof of the fundamental fact that there is a unique signature  $\sigma : S_A \rightarrow F_2$  and that  $\sigma$  is then a group homomorphism.

Let  $\bullet$  be a binary operation on a set  $X$ , written using infix notation, so  $x_1 \bullet x_2$  denotes the product of  $x_1$  and  $x_2$ . The large majority of such operations occurring in mathematics are *associative*, meaning that  $(x_1 \bullet x_2) \bullet x_3 = x_1 \bullet (x_2 \bullet x_3)$  for all  $x_1, x_2, x_3 \in X$ . If  $\bullet$  is associative and  $x_1, x_2, \dots, x_n \in X$  then the product  $x_1 \bullet x_2 \bullet \dots \bullet x_n$  is well-defined, meaning its value does not depend on the order in which the operations are carried out.

This result will be established in Section 9. We first define a particular order of carrying out the operations. This is the order in which at each stage the product of the current first and second components are taken. For example, the product of the 6 components  $x_1, x_2, x_3, x_4, x_5, x_6$  evaluated using this order results in the value  $\bullet(x_1, \dots, x_6) = (((((x_1 \bullet x_2) \bullet x_3) \bullet x_4) \bullet x_5) \bullet x_6)$ . In general, the corresponding product of  $n$  terms will be denoted by  $\bullet(x_1, \dots, x_n)$ . Theorem 9.1 states that if  $\bullet$  is associative then  $\bullet(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = \alpha \bullet \beta$ , where  $\alpha = \bullet(x_1, \dots, x_m)$  and  $\beta = \bullet(x_{m+1}, \dots, x_n)$ . This is a weak form of the generalised associative law, although it is one which is often all that is needed.

Theorem 9.2 gives the general form of the generalised associative law and states that if  $\bullet$  is associative then  $\bullet(x_1, \dots, x_n) = \bullet_{\mathbb{R}}(x_1, \dots, x_n)$  for each  $\mathbb{R}$  from the set of prescriptions describing how the operations are carried out. The main task is to give a rigorous definition of this set. We do this using partitions of intervals of the form  $\{k \in \mathbb{Z} : m \leq k \leq n\}$  in which each element in the partition is also an interval of this form.

By a *partition* of a set  $S$  we mean a subset  $\mathcal{Q}$  of  $\mathcal{P}_0(S)$  such that for each  $s \in S$  there exists a unique  $Q \in \mathcal{Q}$  such that  $s \in Q$ . Thus, different elements in a partition of  $S$  are disjoint and their union is  $S$ .

Consider the product  $((x_1 \bullet x_2) \bullet ((x_3 \bullet x_4) \bullet x_5))$ . The order of operations involved here can be described with the help of the following sequence of partitions of the set  $\{1, 2, 3, 4, 5\}$ :

$\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$   
 $\{\{1\}, \{2\}, \{3, 4\}, \{5\}\}$   
 $\{\{1, 2\}, \{3, 4\}, \{5\}\}$   
 $\{\{1, 2\}, \{3, 4, 5\}\}$   
 $\{\{1, 2, 3, 4, 5\}\}$

For each of these partitions (except the last one) the next partition is obtained by amalgamating two adjacent partitions. Corresponding to these partitions there is a sequence of partial evaluations:

$$\begin{aligned} & \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\} \\ & \{\{x_1\}, \{x_2\}, \{(x_3 \bullet x_4)\}, \{x_5\}\} \\ & \{\{(x_1 \bullet x_2)\}, \{(x_3 \bullet x_4)\}, \{x_5\}\} \\ & \{\{(x_1 \bullet x_2)\}, \{((x_3 \bullet x_4) \bullet x_5)\}\} \\ & \{\{((x_1 \bullet x_2) \bullet ((x_3 \bullet x_4) \bullet x_5))\}\} \end{aligned}$$

and the final expression is essentially the product we started with.

For each finite set  $A$  and each  $B \subset A$  denote the set  $\{C \in \mathcal{P}(A) : C \approx B\}$  by  $A \Delta B$ . The set  $A \Delta B$  plays the role of a *binomial coefficient*: If  $|A| = n$  (with  $|A|$  the usual cardinality of the set  $A$ ) and  $|B| = k$  then  $|A \Delta B| = \binom{n}{k}$ . In Section 10 we establish results which correspond to some of the usual identities for binomial coefficients.

If  $A, B$  and  $C$  are finite sets then we write  $C \approx A \amalg B$  if there exist disjoint sets  $A'$  and  $B'$  with  $A \approx A'$ ,  $B \approx B'$  and  $C \approx A' \cup B'$ . Theorem 10.1 corresponds to the identity  $\binom{n+1}{k+1} \binom{n}{k+1} + \binom{n}{k}$  used to generate Pascal's triangle. It states that if  $A$  is a finite set,  $B$  is a proper subset of  $A$ ,  $a \notin A$  and  $b \in A \setminus B$  then

$$(A \cup \{a\}) \Delta (B \cup \{a\}) \approx (A \Delta B) \amalg (A \Delta (B \cup \{b\})).$$

If  $C$  is a finite set then, as before,  $S_C$  denotes the group of bijections  $h : C \rightarrow C$ . If  $B$  is a subset of a finite set  $A$  then  $I_{B,A}$  denotes the set of injective mappings  $k : B \rightarrow A$ . Theorem 10.2 states that if  $B$  is a subset of a finite set  $A$  then  $I_{B,A} \approx (A \Delta B) \times S_B$ .

Theorem 10.3 states that if  $B$  is a subset of a finite set  $A$  then  $S_A \approx I_{B,A} \times S_{A \setminus B}$ .

Theorem 10.4 corresponds to the usual expression for binomial coefficients:

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}$$

and states that if  $B$  is a subset of a finite set  $A$  then  $S_A \times (A \Delta B) \approx S_B \times S_{A \setminus B}$ .

Theorem 10.5 corresponds to the following identity for binomial coefficients:

$$\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}.$$

It states that if  $A, B$  and  $C$  are finite sets with  $C \subset B \subset A$  then

$$(A \Delta B) \times (B \Delta C) \approx (A \Delta C) \times ((A \setminus C) \Delta (B \setminus C)).$$

In Section 11 we prove Dilworth's decomposition theorem [2] by modifying a proof due to Galvin [3] to work with the present treatment of finite sets. This theorem states that if  $\leq$  is a partial order on a finite set  $A$  then there exists a chain-partition  $\mathcal{C}$  of  $A$  and an antichain  $D$  such that  $D \approx \mathcal{C}$ .

Finally, in Section 12 we give a further characterisation of a set being finite. This can be seen as having something to do with enumerating the elements in the set. Let  $E$  be a set. A subset  $\mathcal{U}$  of  $\mathcal{P}(E)$  is called an *E-selector* if  $\emptyset \in \mathcal{U}$  and for each  $U \in \mathcal{U}^p$  there exists a unique element  $e \in E \setminus U$  such that  $U \cup \{e\} \in \mathcal{U}$ . If  $\mathcal{U}$  is an *E-selector* then a subset  $\mathcal{V}$  of  $\mathcal{U}$  is said to be *invariant* if  $\emptyset \in \mathcal{V}$  and if  $V \cup \{e\} \in \mathcal{V}$  for all  $V \in \mathcal{V}^p$ , where  $e$  is the unique element of  $E \setminus V$  with  $V \cup \{e\} \in \mathcal{U}$ . In other words, a subset  $\mathcal{V}$  of  $\mathcal{U}$  is invariant if and only if it is itself an *E-selector*. An *E-selector*  $\mathcal{U}$  is said to be *minimal* if the only invariant subset of  $\mathcal{U}$  is  $\mathcal{U}$  itself. A minimal *E-selector* containing  $E$  will be called an *E-enumerator*.

Theorem 12.1 states that an *E-enumerator* exists if and only if  $E$  is finite and Theorem 12.2 then shows that if  $A$  is a finite set then an *A-selector* is minimal if and only if it is totally ordered and thus it is an *A-enumerator* if and only if it is totally ordered. This implies that every *A-enumerator* is totally ordered.

For each *A-selector*  $\mathcal{U}$  let  $\mathbf{e}_{\mathcal{U}} : \mathcal{U}^p \rightarrow A$  and  $\mathbf{s}_{\mathcal{U}} : \mathcal{U}^p \rightarrow \mathcal{U} \setminus \{\emptyset\}$  be the mappings with  $\mathbf{e}_{\mathcal{U}}(U) = e$  and  $\mathbf{s}_{\mathcal{U}}(U) = U \cup \{e\}$ , where  $e$  is the unique element in  $A \setminus U$  such that  $U \cup \{e\} \in \mathcal{U}$ . Proposition 12.1 states that if  $\mathcal{U}$  is an *A-enumerator* then the mappings  $\mathbf{s}_{\mathcal{U}} : \mathcal{U}^p \rightarrow \mathcal{U} \setminus \{\emptyset\}$  and  $\mathbf{e}_{\mathcal{U}} : \mathcal{U}^p \rightarrow A$  are both bijections. In particular, if  $\mathcal{U}$  and  $\mathcal{V}$  are *A-enumerators* then  $\mathcal{U} \approx \mathcal{V}$ . (This means, somewhat imprecisely, that any *A-enumerator* contains one more element than  $A$ .)

If  $\mathcal{U}$  is an *A-enumerator* then for each  $U \in \mathcal{U}$  the set  $\mathcal{U} \cap \mathcal{P}(U)$  will be denoted by  $\mathcal{U}_U$ . Note that, as far as the definition of  $\mathcal{U}_U^p$  is concerned,  $\mathcal{U}_U$  is considered here to be a subset of  $\mathcal{P}(U)$  and so  $\mathcal{U}_U^p = \{U' \in \mathcal{U} : U' \text{ is a proper subset of } U\}$ .  $\mathcal{U}_U$  is in fact a *U-enumerator*.

Theorem 12.3 states that if  $\mathcal{U}$  and  $\mathcal{V}$  are any two *A-enumerators* then there exists a unique mapping  $\pi : \mathcal{U} \rightarrow \mathcal{V}$  with  $\pi(\emptyset) = \emptyset$  such that  $\pi(\mathcal{U}^p) \subset \mathcal{V}^p$  and that  $\pi(\mathbf{s}_{\mathcal{U}}(U)) = \mathbf{s}_{\mathcal{V}}(\pi(U))$  for all  $U \in \mathcal{U}^p$ . Moreover, the mapping  $\pi$  is a bijection and  $\pi(A) = A$ .

Now let  $\mathbb{I} = (X, f, x_0)$  be an iterator with the first component  $X$  a set. Then Proposition 12.5 shows that if  $\mathcal{U}$  is an *A-enumerator* then there exists a unique mapping  $\alpha_{\mathcal{U}} : \mathcal{U} \rightarrow X$  with  $\alpha_{\mathcal{U}}(\emptyset) = x_0$  such that  $\alpha_{\mathcal{U}}(\mathbf{s}_{\mathcal{U}}(U)) = f(\alpha_{\mathcal{U}}(U))$  for all  $U \in \mathcal{U}^p$ .

Theorem 12.5 then states that for each *A-enumerator*  $\mathcal{U}$  we have  $\alpha_{\mathcal{U}}(A) = \omega(A)$  where  $\omega$  is the assignment of finite sets in  $\mathbb{I}$ .

We end the Introduction with the following important fact:

**Lemma 1.3** *For each set  $A$  there exists an element  $a$  not in  $A$ . In particular, if  $A$  is finite then by Lemma 1.1  $A \cup \{a\}$  will also be finite.*

*Proof* In fact there must exist an element in  $\mathcal{P}(A) \setminus A$ . If this were not the case then  $\mathcal{P}(A) \subset A$ , and we could define a surjective mapping  $f : A \rightarrow \mathcal{P}(A)$  by letting  $f(x) = x$  if  $x \in \mathcal{P}(A)$  and  $f(x) = \emptyset$  otherwise. But by Cantor's diagonal argument (which states that a mapping  $f : X \rightarrow \mathcal{P}(X)$  cannot be surjective) this is not possible.  $\square$



## 2 Finite sets

Recall that a set  $A$  is defined to be *finite* if  $\mathcal{P}(A)$  is the only inductive  $A$ -system, where a subset  $\mathcal{S}$  of the power set  $\mathcal{P}(A)$  is an inductive  $A$ -system if  $\emptyset \in \mathcal{S}$  and  $B \cup \{a\} \in \mathcal{S}$  for all  $B \in \mathcal{S}^p$ ,  $a \in A \setminus B$  (and where  $\mathcal{S}^p$  denotes the set of subsets in  $\mathcal{S}$  which are proper subsets of  $A$ ).

In this section we establish the basic properties of finite sets. Most of these simply confirm that finite sets are closed under the usual set-theoretic operations. More precisely, if  $A$  and  $B$  are finite sets then their union  $A \cup B$ , their product  $A \times B$  and  $B^A$  (the set of all mappings from  $A$  to  $B$ ) are all finite sets. Moreover, the power set  $\mathcal{P}(A)$  is finite and (Proposition 1.1) any subset of a finite set is finite.

Let us first say something about mappings between finite sets. Let  $f : X \rightarrow Y$  be a mapping (with  $X$  and  $Y$  arbitrary sets). Then the *graph of  $f$*  is the subset  $\Gamma_f = \{(x, y) \in X \times Y : y = f(x) \text{ for some } x \in X\}$  of  $\mathcal{P}(X \times Y)$ . The set  $\Gamma_f$  has the property that for each  $x \in X$  there exists a unique  $y \in Y$  with  $y = f(x)$  and a set with this property will be called an  $X \times Y$ -*graph*. Thus if  $f : X \rightarrow Y$  is a mapping then  $\Gamma_f$  is an  $X \times Y$ -graph. Usually being a graph is taken as the definition of a mapping and so each  $X \times Y$ -graph  $G$  defines a mapping  $f : X \rightarrow Y$  with of course  $\Gamma_f = G$ . But if  $A$  and  $B$  are finite sets and with the definition of being finite being used here we can be much more explicit about defining mappings from  $A$  to  $B$ . We assume that the following statements are valid for the mappings to be considered here: (The sets occurring below are all finite.)

- (1) If  $f : A \rightarrow B$  is a mapping then to each  $a \in A$  there is associated a unique element  $f(a)$  of  $B$  (the value of  $f$  at  $a$ ). In particular, this implies that the set  $\Gamma_f = \{(a, b) \in A \times B : b = f(a) \text{ for some } a \in A\}$  is an  $A \times B$ -graph. It also implies that if  $A$  is non-empty then there can be no mapping  $f : A \rightarrow \emptyset$ .
- (2) Mappings are determined by their values, meaning that if  $f, g : A \rightarrow B$  are mappings with  $f(a) = g(a)$  for all  $a \in A$  then  $f = g$ . Equivalently, if  $\Gamma_f = \Gamma_g$  then  $f = g$ .
- (3) For each set  $A$  there is the identity mapping  $\text{id}_A : A \rightarrow A$  satisfying  $\text{id}_A(a) = a$  for all  $a \in A$ ; these mappings are bijections. Note that  $\text{id}_\emptyset : \emptyset \rightarrow \emptyset$  is the unique mapping  $f : \emptyset \rightarrow \emptyset$  (since mappings are determined by their values).
- (4) If  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are two mappings then there is a mapping  $g \circ f : A \rightarrow C$  (their composition) satisfying  $(g \circ f)(a) = g(f(a))$  for all  $a \in A$ .
- (5) If  $A_1$  and  $A_2$  are disjoint and  $f_1 : A_1 \rightarrow B$  and  $f_2 : A_2 \rightarrow B$  are mappings then there is a mapping  $f : A_1 \cup A_2 \rightarrow B$  satisfying  $f(a_1) = f_1(a_1)$  if  $a_1 \in A_1$  and  $f(a_2) = f_2(a_2)$  if  $a_2 \in A_2$ . In particular, if  $f : A \rightarrow B$  is a mapping and  $a \notin A$

then  $f$  can be extended to a mapping  $f' : A \cup \{a\} \rightarrow B$  with  $f'(a)$  chosen to be any element in  $B$ .

(6) If  $f : A \rightarrow B$  is a mapping and  $C \subset A$  then there is the restriction mapping  $f|_C : C \rightarrow B$  satisfying  $f|_C(c) = f(c)$  for all  $c \in C$ .

(7) If  $f : A \rightarrow B$  is a mapping and  $C$  is a finite set with  $f(A) \subset C$  then there is the extension mapping  $f|_C : A \rightarrow C$  satisfying  $f|_C(a) = f(a)$  for all  $a \in A$ . (If  $f : A \rightarrow B$  and  $A' \subset A$  then as usual the set  $\{f(c) : c \in A'\}$  is denoted by  $f(A')$ .) In particular, for each finite set  $B$  there is a unique mapping  $f_\emptyset^B : \emptyset \rightarrow B$ . This mapping is unique since mappings are determined by their values.

(8) If  $f_1 : A_1 \rightarrow B_1$  and  $f_2 : A_2 \rightarrow B_2$  are mappings then there is a mapping  $g : A_1 \times B_1 \rightarrow A_2 \times B_2$  satisfying  $g((a, b) = (f_1(a), f_2(a))$  for all  $(a, b) \in A_1 \times B_1$ .

(9) A mapping can be defined by a formula involving a finite number of cases (where finite means an explicit number such as two or three). For example, if  $b, c \in E$  with  $b \neq c$  then a transposition  $\tau : E \rightarrow E$  can be defined by

$$\tau(x) = \begin{cases} c & \text{if } x = b, \\ b & \text{if } x = c, \\ x & \text{otherwise.} \end{cases}$$

(10) The formula in (9) can also involve a mapping which has already been defined. For example, if  $a \notin A$ ,  $A'$  is a non-empty subset of  $A$  and  $f : A \cup \{a\} \rightarrow B$  has been defined previously then a new mapping  $g : A \rightarrow B$  can be defined by

$$g(c) = \begin{cases} f(c) & \text{if } c \in A \setminus A', \\ f(a) & \text{if } c \in A'. \end{cases}$$

We will see later in Proposition 2.6 that if  $G$  is an  $A \times B$ -graph then there is a mapping  $f : A \rightarrow B$  obtained using only the above statements such that  $\Gamma_f = G$ . Thus for finite sets being a graph could be taken as the definition of a mapping.

We start by looking at a fundamental property which depends crucially on the set involved being finite. One reason for presenting the result at this point is to convince the reader that the definition of being finite employed here leads to rather straightforward proofs.

**Theorem 2.1** *Let  $A$  be finite and  $f : A \rightarrow A$  be a mapping. Then  $f$  is injective if and only if it is surjective (and thus if and only if it is bijective).*

*Proof* We first show that an injective mapping is bijective. Let  $\mathcal{S}$  be the set consisting of those  $B \in \mathcal{P}(A)$  having the property that every injective mapping  $p : B \rightarrow B$  is bijective. Then  $\emptyset \in \mathcal{S}$ , since the only mapping  $p : \emptyset \rightarrow \emptyset$  is bijective. Let  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ ; consider an injective mapping  $p : B \cup \{a\} \rightarrow B \cup \{a\}$ . There are two cases:

( $\alpha$ )  $p(B) \subset B$ . Then the restriction  $p|_B : B \rightarrow B$  of  $p$  to  $B$  is injective and hence bijective, since  $B \in \mathcal{S}$ . If  $p(a) \in B$  then  $p(b) = p|_B(b) = p(a)$  for some  $b \in B$ , since  $p|_B$  is surjective, which contradicts the fact that  $p$  is injective. Thus  $p(a) = a$ , and it follows that  $p$  is bijective.

( $\beta$ )  $p(B) \not\subset B$ . In this case there exists  $b \in B$  with  $p(b) = a$  and, since  $p$  is injective, we must have  $p(c) \in B$  for all  $c \in B \setminus \{b\}$  and  $p(a) \in B$ . This means there is an injective mapping  $q : B \rightarrow B$  defined by letting

$$q(c) = \begin{cases} p(c) & \text{if } c \in B \setminus \{b\}, \\ p(a) & \text{if } c = b \end{cases}$$

and then  $q$  is bijective, since  $B \in \mathcal{S}$ . Therefore  $p$  is again bijective.

This shows that  $B \cup \{a\} \in \mathcal{S}$  and thus that  $\mathcal{S}$  is an inductive  $A$ -system. Hence  $\mathcal{S} = \mathcal{P}(A)$ , since  $A$  is finite. In particular,  $A \in \mathcal{S}$  and so every injective mapping  $f : A \rightarrow A$  is bijective.

We now show that a surjective mapping is bijective, and here let  $\mathcal{S}$  be the set consisting of those elements  $B \in \mathcal{P}(A)$  having the property that every surjective mapping  $p : B \rightarrow B$  is bijective. Then  $\emptyset \in \mathcal{S}$ , again since the only mapping  $p : \emptyset \rightarrow \emptyset$  is bijective. Let  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ ; consider a surjective mapping  $p : B \cup \{a\} \rightarrow B \cup \{a\}$ . Let  $D = \{b \in B : p(b) = a\}$ ; there are three cases:

( $\alpha$ )  $D = \emptyset$ . Then  $p(a) = a$ , since  $p$  is surjective, thus the restriction  $p|_B : B \rightarrow B$  of  $p$  to  $B$  is surjective and hence bijective (since  $B \in \mathcal{S}$ ), and this means  $p$  is bijective.

( $\beta$ )  $D \neq \emptyset$  and  $p(a) \in B$ . Here we can define a surjective mapping  $q : B \rightarrow B$  by letting

$$q(c) = \begin{cases} p(c) & \text{if } c \in B \setminus D, \\ p(a) & \text{if } c \in D. \end{cases}$$

Thus  $q$  is bijective (since  $B \in \mathcal{S}$ ), which implies that  $D = \{b\}$  for some  $b \in C$  and in particular  $p$  is also injective.

( $\gamma$ )  $D \neq \emptyset$  and  $p(a) = a$ . This is not possible since then  $p(B \setminus D) = B$  and so, choosing any  $b \in D$ , the mapping  $h : B \rightarrow B$  with

$$q(c) = \begin{cases} p(c) & \text{if } c \in B \setminus D, \\ b & \text{if } c \in D \end{cases}$$

would be surjective but not injective (since there also exists  $c \in B \setminus D$  with  $p(c) = b$ ).

This shows that  $B \cup \{a\} \in \mathcal{S}$  and thus that  $\mathcal{S}$  is an inductive  $A$ -system. Hence  $\mathcal{S} = \mathcal{P}(A)$ , since  $A$  is finite. In particular,  $A \in \mathcal{S}$  and so every surjective mapping  $f : A \rightarrow A$  is bijective.  $\square$

Note that Theorem 2.1 implies the set  $\mathbb{N}$  of natural numbers is infinite (i.e., it is not finite), since the successor mapping  $s : \mathbb{N} \rightarrow \mathbb{N}$  with  $s(n) = n + 1$  for all  $n \in \mathbb{N}$  is injective but not surjective.

If  $E, F$  are any sets then we write  $E \approx F$  if there exists a bijective mapping  $f : E \rightarrow F$ . The following result is a direct corollary of Theorem 2.1:

**Theorem 2.2** *If  $B$  is a subset of a finite set  $A$  with  $B \approx A$  then  $B = A$ .*

*Proof* There exists a bijective mapping  $f : A \rightarrow B$  and the restriction  $f|_B : B \rightarrow B$  of  $f$  to  $B$  is then injective; thus by Theorem 2.1  $f|_B$  is bijective. But this is only possible if  $B = A$ , since if  $a \in A \setminus B$  then  $f(a) \notin f|_B(B)$ .  $\square$

The form of the proof of Theorem 2.1 is repeated in practically every proof which follows: In general we will start with some statement  $P$  about finite sets, meaning for each finite set  $A$  we have a statement  $P(A)$ . (For example,  $P(A)$  could be the statement that any injective mapping  $f : A \rightarrow A$  is surjective.) The aim is then to establish that  $P$  is a property of finite sets, i.e., to establish that  $P(A)$  holds for every finite set  $A$ . To accomplish this we fix a finite set  $A$  and consider the set  $\mathcal{S} = \{B \in \mathcal{P}(A) : P(B) \text{ holds}\}$  (recalling from Proposition 1.1 that each subset of  $A$  is finite). We then show that  $\mathcal{S}$  is an inductive  $A$ -system (i.e., show that  $P(\emptyset)$  holds and  $P(B \cup \{a\})$  holds whenever  $B \in \mathcal{S}$  and  $a \in A \setminus B$ ) to conclude that  $\mathcal{S} = \mathcal{P}(A)$ , since  $A$  is finite. In particular  $A \in \mathcal{S}$ , i.e.,  $P(A)$  holds.

This template for proving facts about finite sets can be regarded as a ‘local’ version of the following *induction principle for finite sets* which first appeared in a 1909 paper of Zermelo [12]:

**Theorem 2.3** *Let  $P$  be a statement about finite sets. Suppose  $P(\emptyset)$  holds and that  $P(A \cup \{a\})$  holds for each element  $a \notin A$  whenever  $P(A)$  holds for a finite set  $A$ . Then  $P$  is a property of finite sets, i.e.,  $P(A)$  holds for every finite set  $A$ .*

*Proof* Let  $A$  be a finite set and recall from Proposition 1.1 that each subset of  $A$  is finite. Put  $\mathcal{S} = \{B \in \mathcal{P}(A) : P(B) \text{ holds}\}$ ; then  $\mathcal{S}$  is an inductive  $A$ -system and hence  $\mathcal{S} = \mathcal{P}(A)$ . In particular  $A \in \mathcal{S}$ , i.e.,  $P(A)$  holds.  $\square$

The above proof of Theorem 2.1 and nearly all the proofs which follow can easily be converted into proofs based on Theorem 2.3. As an example, we give two proofs of Proposition 2.1 below. Proofs based on Theorem 2.3 seem to be more elegant (although this really a matter of taste). However, we prefer to continue with the style used in the proof of Theorem 2.1, since such proofs are internal to the finite set being considered, and thus appear to be more concrete. These proofs almost always end with a mantra of the form:

*It follows that  $\mathcal{S}$  is an inductive  $A$ -system. Thus  $\mathcal{S} = \mathcal{P}(A)$ , since  $A$  is finite. In particular,  $A \in \mathcal{S}$  and so the statement about  $A$  holds.*

and this will be shortened to the following:

*It follows that  $\mathcal{S}$  is an inductive  $A$ -system. Thus  $A \in \mathcal{S}$  and so the statement about  $A$  holds.*

We now establish the usual properties of finite sets mentioned above. The proofs are mostly very straightforward and, since they all follow the same pattern, they tend to become somewhat monotonous.

**Proposition 2.1** *If  $A$  and  $B$  are finite sets then so is  $A \cup B$ .*

*Proof* Consider the set  $\mathcal{S} = \{C \in \mathcal{P}(A) : C \cup B \text{ is finite}\}$ . Then  $\emptyset \in \mathcal{S}$ , since by assumption  $\emptyset \cup B = B$  is finite and if  $C \in \mathcal{S}$  (i.e.,  $C \cup B$  is finite) and  $a \in A \setminus C$  then by Lemma 1.1  $(C \cup \{a\}) \cup B = (C \cup B) \cup \{a\} \in \mathcal{S}$ . It follows that  $\mathcal{S}$  is an inductive  $A$ -system. Thus  $A \in \mathcal{S}$  and so  $A \cup B$  is finite.

Here is a proof based on Theorem 2.3: Consider the finite set  $B$  to be fixed and for each finite set  $A$  let  $P(A)$  be the statement that  $A \cup B$  is finite. Then  $P(\emptyset)$  holds, since by assumption  $\emptyset \cup B = B$  is finite. Moreover, if  $P(A)$  holds (i.e.,  $A \cup B$  is finite) and  $a \notin A$  then by Lemma 1.1  $(A \cup \{a\}) \cup B = (A \cup B) \cup \{a\}$  is finite, i.e.,  $P(A \cup \{a\})$  holds. Thus by Theorem 2.3  $A \cup B$  is finite for every finite set  $A$ .  $\square$

**Proposition 2.2** *Let  $A$  and  $E$  be sets with  $A$  finite.*

- (1) *If there exists an injective mapping  $f : E \rightarrow A$  then  $E$  is also finite.*
- (2) *If there exists a surjective mapping  $f : A \rightarrow E$  then  $E$  is again finite.*
- (3) *If  $E$  and  $F$  are any sets with  $E \approx F$  then  $E$  is finite if and only if  $F$  is.*

*Proof* (1) Let  $\mathcal{S}$  be the set consisting of those elements  $C \in \mathcal{P}(A)$  such that if  $D$  is any set for which there exists an injective mapping  $p : D \rightarrow C$  then  $D$  is finite. Then  $\emptyset \in \mathcal{S}$ , since there can only exist a mapping  $p : D \rightarrow \emptyset$  if  $D = \emptyset$  and the empty set  $\emptyset$  is finite. Let  $C \in \mathcal{S}^p$  and  $a \in A \setminus C$ . Consider a set  $D$  for which there exists an injective mapping  $p : D \rightarrow C \cup \{a\}$ . There are two cases:

( $\alpha$ )  $p(d) \in C$  for all  $d \in D$ . Here we can consider  $p$  as a mapping from  $D$  to  $C$  and as such it is still injective. Thus  $D$  is finite since  $C \in \mathcal{S}$ .

( $\beta$ ) There exists an element  $b \in D$  with  $p(b) = a$ . Put  $D' = D \setminus \{b\}$ . Now since  $p$  is injective it follows that  $p(d) \neq a$  for all  $d \in D'$ , and thus we can define a mapping  $q : D' \rightarrow C$  by letting  $q(d) = p(d)$  for all  $d \in D'$ . Then  $q : D' \rightarrow C$  is also injective (since  $p : D \rightarrow C \cup \{a\}$  is) and therefore  $D'$  is finite since  $C \in \mathcal{S}$ . Hence by Lemma 1.1  $D = D' \cup \{b\}$  is finite.

This shows that  $C \cup \{a\} \in \mathcal{S}$  and therefore  $\mathcal{S}$  is an inductive  $A$ -system. Thus  $A \in \mathcal{S}$ , which means that if there exists an injective mapping  $f : E \rightarrow A$  then  $E$  is also finite.

(2) Let  $\mathcal{S}$  be the set consisting of those elements  $C \in \mathcal{P}(A)$  such that if  $D$  is any set for which there exists a surjective mapping  $p : C \rightarrow D$  then  $D$  is finite. Then  $\emptyset \in \mathcal{S}$ , since there can only exist a surjective mapping  $p : \emptyset \rightarrow D$  if  $D = \emptyset$  and the empty set  $\emptyset$  is finite. Let  $C \in \mathcal{S}^p$  and  $a \in A \setminus C$ . Consider a set  $D$  for which there exists a surjective mapping  $p : C \cup \{a\} \rightarrow D$ . There are again two cases:

( $\alpha$ ) The restriction  $p|_C : C \rightarrow D$  of  $p$  to  $C$  is still surjective. Then  $D$  is finite since  $C \in \mathcal{S}$ .

( $\beta$ ) The restriction  $p|_C$  is not surjective. Put  $b = p(a)$  and  $D' = D \setminus \{b\}$ . Then  $p(c) \neq b$  for all  $c \in C$  (since  $p|_C$  is not surjective) and therefore we can define a mapping  $q : C \rightarrow D'$  by letting  $q(c) = p(c)$  for all  $c \in C$ . But  $p : C \cup \{a\} \rightarrow D$  is surjective and hence  $q : C \rightarrow D'$  is also surjective. Thus  $D'$  is finite since  $C \in \mathcal{S}$  holds, and so by Lemma 1.1  $D = D' \cup \{b\}$  is finite.

This shows that  $C \cup \{a\} \in \mathcal{S}$  and it follows that  $\mathcal{S}$  is an inductive  $A$ -system. Thus  $A \in \mathcal{S}$ , which means that if there exists a surjective mapping  $f : A \rightarrow E$  then  $E$  is also finite.

(3) This is now clear.  $\square$

**Proposition 2.3** *If  $A$  is a finite set then so is the power set  $\mathcal{P}(A)$ .*

*Proof* Let  $\mathcal{S}$  be the set consisting of those elements  $B \in \mathcal{P}(A)$  for which the power set  $\mathcal{P}(B)$  is finite. Then by Lemma 1.1  $\emptyset \in \mathcal{S}$ , since  $\mathcal{P}(\emptyset) = \{\emptyset\} = \emptyset \cup \{\emptyset\}$ . Thus consider  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ . Then  $\mathcal{P}(B \cup \{a\}) = \mathcal{P}(B) \cup \mathcal{P}_a(B)$ , where

$\mathcal{P}_a(B) = \{C \cup \{a\} : C \in \mathcal{P}(B)\}$ , and the mapping  $C \mapsto C \cup \{a\}$  from  $\mathcal{P}(B)$  to  $\mathcal{P}_a(B)$  is surjective. It follows from Proposition 2.2 (2) that  $\mathcal{P}_a(B)$  is finite and so by Proposition 2.1  $\mathcal{P}(B \cup \{a\})$  is finite, i.e.,  $B \cup \{a\} \in \mathcal{S}$ . This shows that  $\mathcal{S}$  is an inductive  $A$ -system. Hence  $A \in \mathcal{S}$  and so the power set  $\mathcal{P}(A)$  is finite.  $\square$

**Proposition 2.4** *If  $A$  and  $B$  are finite sets then so is their product  $A \times B$ .*

*Proof* Consider  $\mathcal{S} = \{C \in \mathcal{P}(A) : C \times B \text{ is finite}\}$ . Then  $\emptyset \in \mathcal{S}$ , since  $\emptyset \times B = \emptyset$ . Let  $C \in \mathcal{S}$  and  $a \in A \setminus C$ . Then  $(C \cup \{a\}) \times B = (C \times B) \cup (\{a\} \times B)$  and by Proposition 2.2 (2)  $\{a\} \times B$  is finite since the mapping  $f : B \rightarrow \{a\} \times B$  with  $f(b) = (a, b)$  for all  $b \in B$  is surjective. Thus by Proposition 2.1  $(C \cup \{a\}) \times B$  is finite, i.e.,  $C \cup \{a\} \in \mathcal{S}$ . This shows that  $\mathcal{S}$  is an inductive  $A$ -system. Hence  $A \in \mathcal{S}$  and so  $A \times B$  is finite.  $\square$

**Proposition 2.5** *If  $A$  and  $B$  are finite sets then so is  $B^A$ , the set of all mappings from  $A$  to  $B$ .*

*Proof* Define a mapping  $\gamma : B^A \rightarrow \mathcal{P}(A \times B)$  by letting

$$\gamma(f) = \{(a, b) \in A \times B : b = f(a) \text{ for some } a \in A\}.$$

Let  $f, g \in B^A$  with  $\gamma(f) = \gamma(g)$  and let  $a \in A$ . Then  $(a, f(a)) \in \gamma(f)$  and so  $(a, f(a)) = (a', g(a'))$  for some  $a' \in A$ , since  $\gamma(f) = \gamma(g)$ . Thus  $a = a'$  and  $f(a) = g(a') = g(a)$  and so  $f(a) = g(a)$  for all  $a \in A$ , i.e.,  $f = g$ . This shows that  $\gamma$  is injective and hence by Propositions 2.1, 2.2 (1) and 2.3  $B^A$  is finite.  $\square$

Note that in this proof the only property of  $B^A$  that has been used is that mappings are determined by their values, meaning that if  $f, g \in B^A$  with  $f(a) = g(a)$  for all  $a \in A$  then  $f = g$ .

Recall that if  $A$  and  $B$  are finite sets then  $G \subset \mathcal{P}(A \times B)$  is an  $A \times B$ -graph if for each  $a \in A$  there exists a unique  $b \in B$  with  $(a, b) \in G$ . In particular, if  $f : A \rightarrow B$  is a mapping then the set  $\Gamma_f = \{(a, b) \in A \times B : b = f(a) \text{ for some } a \in A\}$  is an  $A \times B$ -graph.

**Proposition 2.6** *Let  $A$  and  $B$  be finite sets. Then for each  $A \times B$ -graph  $G$  there exists a unique mapping  $f : A \rightarrow B$  such that  $G = \Gamma_f$ .*

*Proof* The uniqueness is clear since mappings are determined by their values. For the existence consider the subset  $\mathcal{S}$  of  $\mathcal{P}(A)$  consisting of those  $C \subset A$  having the property that for each  $C \times B$ -graph  $G$  there exists a mapping  $f : C \rightarrow B$  with  $\Gamma_f = G$ . Then  $\emptyset \in \mathcal{S}$ , since  $\emptyset$  is an  $\emptyset \times B$ -graph, there is a mapping  $p : \emptyset \rightarrow B$  and  $\emptyset = \Gamma_p$ .

Thus let  $C \in \mathcal{S}$ ,  $c \in A \setminus \{c\}$ , put  $C' = C \cup \{c\}$  and let  $G$  be a  $C' \times B$ -graph. Then  $G' = G \cap \mathcal{P}(C \times B)$  is a  $C \times B$ -graph and so there exists a mapping  $h : C \rightarrow B$  with  $G' = \Gamma_h$ , since  $C \in \mathcal{S}$ . Extend  $h$  to a mapping  $h' : C' = C \cup \{c\} \rightarrow B$  by letting  $h'(c) = d$ , where  $d$  is the unique element of  $B$  with  $(c, d) \in G$ . Then  $\Gamma_{h'} = G$ , which shows that  $C' \in \mathcal{S}$ . Thus  $\mathcal{S}$  is an inductive  $A$ -system and hence  $A \in \mathcal{S}$ . Therefore for each  $A \times B$ -graph  $G$  there exists a mapping  $f : A \rightarrow B$  such that  $G = \Gamma_f$ .  $\square$

**Proposition 2.7** *Let  $A$ ,  $B$  and  $C$  be finite sets, let  $f : C \rightarrow A$  be a surjective mapping and let  $g : C \rightarrow B$  be a mapping. Then there exists a mapping  $h : A \rightarrow B$  with  $g = h \circ f$  if and only if  $g(c) = g(c')$  whenever  $c, c' \in C$  with  $f(c) = f(c')$ . Moreover, if  $h$  exists then it is unique.*

*Proof* Suppose first that there exists  $f : A \rightarrow B$  with  $g = h \circ f$ . If  $c, c' \in C$  with  $f(c) = f(c')$  then  $g(c) = h(f(c)) = h(f(c')) = g(c')$  and so  $g(c) = g(c')$  whenever  $f(c) = f(c')$ . Moreover,  $h(f(c)) = g(c)$  for each  $c \in C$  and  $f$  is surjective and hence  $h$  is uniquely determined by  $f$  and  $g$ .

Now suppose that  $g(c) = g(c')$  whenever  $c, c' \in C$  are such that  $f(c) = f(c')$ . Let

$$G = \{(a, b) \in A \times B : \text{there exists } c \in C \text{ with } a = f(c) \text{ and } b = g(c)\}.$$

Let  $a \in A$ ; then  $a = f(c)$  for some  $c \in C$ , since  $f$  is surjective and then  $(a, b) \in G$  with  $b = g(c)$ . For each  $a \in A$  there thus exists at least one  $b \in B$  with  $(a, b) \in G$ . But if also  $(a, b') \in G$  then there exists  $c' \in C$  with  $a = f(c')$  and  $b' = g(c')$ . In particular  $f(c) = f(c')$  and so  $b = g(c) = g(c') = b'$ , i.e.,  $b = b'$ . Hence for each  $a \in A$  there exists a unique  $b \in B$  with  $(a, b) \in G$ , which shows that  $G$  is an  $A \times B$ -graph. Therefore by Proposition 2.6 there exists a unique mapping  $h : A \rightarrow B$  such that  $G = \Gamma_h$ . This means that  $G = \{(a, b) \in A \times B : h(a) = b\}$ . Let  $c \in C$ ; then  $(f(c), h(f(c))) \in G$ . But also  $(f(c), g(c)) \in G$  and there is a unique  $b \in B$  such that  $(f(c), b) \in G$ . Hence  $g(c) = h(f(c))$  and therefore  $g = h \circ f$ .  $\square$

**Proposition 2.8** *Let  $A$  and  $B$  be finite sets and let  $f : A \rightarrow B$  be a bijection. Then there exists a unique mapping  $f^{-1} : B \rightarrow A$  such that  $f^{-1} \circ f = \text{id}_A$  and  $f \circ f^{-1} = \text{id}_B$ . Moreover,  $f^{-1}$  is a bijection.*



*Proof* Let  $\mathcal{S}$  denote the set of subsets  $C$  of  $A$  for which there exists a unique mapping  $f|_C^{-1} : f(C) \rightarrow C$  such that  $f|_C \circ f|_C^{-1} = \text{id}_{f(C)}$  and  $f|_C^{-1} \circ f|_C = \text{id}_C$  and so in particular  $\emptyset \in \mathcal{S}$ . Let  $C \in \mathcal{S}^p$  and  $a \in A \setminus C$ ; put  $C' = C \cup \{a\}$ . We have the unique mapping  $f|_C^{-1} : f(C) \rightarrow C$  such that  $f|_C \circ f|_C^{-1} = \text{id}_{f(C)}$  and  $f|_C^{-1} \circ f|_C = \text{id}_C$  and can define  $f|_{C'}^{-1} : f(C') \rightarrow C'$  by letting  $f|_{C'}^{-1}(d) = f|_C^{-1}(d)$  if  $d \in f(C)$  and putting  $f|_{C'}^{-1}(f(a)) = a$ . Then  $f|_{C'}^{-1} : f(C') \rightarrow C'$  is the unique mapping such that  $f|_{C'} \circ f|_{C'}^{-1} = \text{id}_{f(C')}$  and  $f|_{C'}^{-1} \circ f|_{C'} = \text{id}_{C'}$  and hence  $C \cup \{a\} \in \mathcal{S}$ . Thus  $\mathcal{S}$  is an inductive  $A$ -system, and therefore  $\mathcal{S} = \mathcal{P}(A)$ , since  $A$  is finite. In particular,  $A \in \mathcal{S}$ . This shows that there exists a unique mapping  $f^{-1} : B \rightarrow A$  such that  $f^{-1} \circ f = \text{id}_A$  and  $f \circ f^{-1} = \text{id}_B$ . It is clear that  $f^{-1}$  is a bijection.  $\square$

Proposition 2.8 also follows from Proposition 2.6. If  $f : A \rightarrow B$  is a bijection then for each  $b \in B$  there exists a unique  $a \in A$  with  $(a, b) \in \Gamma_f$ . Therefore the set  $G = \{(b, a) \in B \times A : (a, b) \in \Gamma_f\}$  is a  $B \times A$ -graph and thus by Proposition 2.6 there exists a unique mapping  $g : B \rightarrow A$  with  $\Gamma_g = G$  and  $g$  is a bijection. Let  $a \in A$ ; then  $(a, f(a)) \in \Gamma_f$  and so  $(f(a), a) \in \Gamma_g$ . Thus  $g(f(a)) = a$ , i.e.,  $g \circ f = \text{id}_A$ . Let  $b \in B$ ; then  $(b, g(b)) \in \Gamma_g$  and so  $(g(b), b) \in \Gamma_f$ . Thus  $f(g(b)) = b$ , i.e.,  $f \circ g = \text{id}_B$ . Now  $g(f(a)) = a$  for all  $a \in A$  and  $f$  is a bijection and hence  $g$  is uniquely determined by  $f$ . This shows that  $g = f^{-1}$ .

The next result holds for arbitrary sets  $E$  and  $F$ , the second statement then being the Cantor-Bernstein-Schröder theorem. (The first statement only holds in general assuming the axiom of choice.) As can be seen, the proofs for finite sets are trivial in comparison to those for the general case.

**Theorem 2.4** *Let  $A$  and  $B$  be finite sets. Then either there exists an injective mapping  $f : A \rightarrow B$  or an injective mapping  $g : B \rightarrow A$ . Moreover, if there exists both an injective mapping  $f : A \rightarrow B$  and an injective mapping  $g : B \rightarrow A$  then  $A \approx B$ .*

*Proof* Let  $\mathcal{S}$  be the set consisting of those  $C \in \mathcal{P}(A)$  for which there either there exists an injective mapping  $p : C \rightarrow B$  or an injective mapping  $q : B \rightarrow C$ . Then  $\emptyset \in \mathcal{S}$ , since the only mapping  $p : \emptyset \rightarrow B$  is injective. Let  $C \in \mathcal{S}^p$  and let  $a \in A \setminus C$ . There are two cases:

( $\alpha$ ) There exists an injective mapping  $q : B \rightarrow C$ . Then  $q$  is still injective when considered as a mapping from  $B$  to  $C \cup \{a\}$ .

( $\beta$ ) There exists an injective mapping  $p : C \rightarrow B$ . If  $p$  is not surjective then it can be extended to an injective mapping  $p' : C \cup \{a\} \rightarrow B$  (with  $p'(a)$  chosen to be any element in  $B \setminus f(C)$ ). On the other hand, if  $p$  is surjective (and hence a bijection)

then the inverse mapping  $p^{-1} : B \rightarrow C$  given in Proposition 2.8 is injective and so is still injective when considered as a mapping from  $B$  to  $C \cup \{a\}$ .

This shows that  $B \cup \{a\} \in \mathcal{S}$  and thus that  $\mathcal{S}$  is an inductive  $A$ -system. Hence  $A \in \mathcal{S}$  and so there either exists an injective mapping  $f : A \rightarrow B$  or an injective mapping  $g : B \rightarrow A$ .

Suppose there exists both an injective mapping  $f : A \rightarrow B$  and an injective mapping  $g : B \rightarrow A$ . Then  $f \circ g : B \rightarrow B$  is an injective mapping, which by Theorem 2.1 is bijective. In particular  $f$  is surjective and therefore bijective, i.e.,  $A \approx B$ .  $\square$

For sets  $E$  and  $F$  we write  $E \preceq F$  if there exists an injective mapping  $f : E \rightarrow F$ . Theorem 2.4 thus states that if  $A$  and  $B$  are finite sets then either  $A \preceq B$  or  $B \preceq A$  and, moreover, that  $A \preceq B$  and  $B \preceq A$  both hold if and only if  $A \approx B$ .

**Lemma 2.1** *Let  $A$  and  $B$  be finite sets with  $B \preceq A$ . Then there exists a subset  $B'$  of  $A$  with  $B' \approx B$ .*

*Proof* There exists an injective mapping  $g : B \rightarrow A$ . Put  $B' = g(B)$ ; then  $B' \subset A$  with  $B' \approx B$  (since  $g$  as a mapping from  $B$  to  $B'$  is a bijection).  $\square$

**Lemma 2.2** *Let  $A$  and  $B$  be finite sets. Then there exists either a subset  $B'$  of  $A$  with  $B' \approx B$  or a subset  $A'$  of  $B$  with  $A' \approx A$ . Moreover, if  $A \not\approx B$  then there exists either a proper subset  $B'$  of  $A$  with  $B' \approx B$  or a proper subset  $A'$  of  $B$  with  $A' \approx A$ .*

*Proof* By Theorem 2.4 either  $A \preceq B$  or  $B \preceq A$ . Thus by Lemma 2.1 there exists either a subset  $B'$  of  $A$  with  $B' \approx B$  or a subset  $A'$  of  $B$  with  $A' \approx A$ . The final statement now follows from Theorem 2.2.  $\square$

**Lemma 2.3** *Let  $A'$  and  $B'$  be finite sets. Then there exist finite sets  $A$  and  $B$  with  $A \approx A'$  and  $B \approx B'$  and either  $B \subset A$  or  $A \subset B$ .*

*Proof* This follows directly from Lemma 2.2.  $\square$

**Lemma 2.4** *Let  $A$  and  $E$  be sets with  $A$  finite. Then there exists a set  $A'$  disjoint from  $E$  with  $A \approx A'$ .*

*Proof* Let  $\mathcal{S}$  be the set of subsets  $B \in \mathcal{P}(A)$  for which there exists a set  $B'$  disjoint from  $E$  with  $B \approx B'$ , and so  $\emptyset \in \mathcal{S}$ . Thus let  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ . Let  $B'$  be disjoint from  $E$  with  $B \approx B'$ . By Lemma 1.3 there exists an element  $b$  not in  $E \cup B'$ . Then  $B' \cup \{b\}$  is disjoint from  $E$  and  $B \cup \{a\} \approx B' \cup \{b\}$  and hence  $B \cup \{a\} \in \mathcal{S}$ . This shows that  $\mathcal{S}$  is an inductive- $A$ -system. Therefore  $A \in \mathcal{S}$ , i.e., there exists a set  $A'$  disjoint from  $E$  with  $A \approx A'$ .  $\square$

**Proposition 2.9** *Let  $A_1$  and  $A_2$  be finite sets. Then there exist disjoint sets  $B_1$  and  $B_2$  with  $B_1 \approx A_1$  and  $B_2 \approx A_2$ .*

*Proof* This follows immediately from Lemma 2.4.  $\square$

The next result can be seen as a version of the Axiom of Choice for finite sets.

**Proposition 2.10** *Let  $A$  and  $A'$  be finite sets and  $f : A \rightarrow A'$  be a surjective mapping. Then there exists  $C \subset A$  such that the restriction  $f|_C : C \rightarrow A'$  is a bijection.*

*Proof* Let  $\mathcal{S}$  be the set of subsets  $B$  of  $A$  such that if  $p : B \rightarrow B'$  is a surjective mapping then there exists  $D \subset B$  such that the restriction  $p|_D : D \rightarrow B'$  is a bijection. Then  $\emptyset \in \mathcal{S}$  and so let  $B \in \mathcal{S}^p$ ,  $b \in A \setminus B$  and  $p : B \cup \{b\} \rightarrow B'$  be a surjective mapping. Then  $p|_B : B \rightarrow B' \setminus \{p(b)\}$  is surjective and so there exists  $D \subset B$  such that  $p|_D : D \rightarrow B' \setminus \{p(b)\}$  is a bijection, since  $B \in \mathcal{S}$ . Put  $D' = D \cup \{b\}$ ; then  $D' \subset B \cup \{b\}$  and  $p|_{D'} : D' \rightarrow B'$  is a bijection. Thus  $B \cup \{b\} \in \mathcal{S}$  and hence  $\mathcal{S}$  is an inductive  $A$ -system. This shows that if  $f : A \rightarrow A'$  is a surjective mapping then there exists  $C \subset A$  such that the restriction  $f|_C : C \rightarrow A'$  is a bijection.  $\square$

**Lemma 2.5** *Let  $A$  and  $B$  be finite sets and  $f : A \rightarrow B$  be a surjective mapping. Then there exists a mapping  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$ . (The mapping  $g$  is clearly injective.)*

*Proof* By Proposition 2.10 there exists  $C \subset A$  such that  $f|_C : C \rightarrow B$  is a bijection and by Proposition 2.8 there is the inverse mapping  $h : B \rightarrow C$  which we can consider as a mapping  $g : B \rightarrow A$ . Clearly  $f \circ g = \text{id}_B$ .  $\square$

The next result is a kind of cancellation law for finite sets.

**Proposition 2.11** (1) If  $A_1$ ,  $A_2$  and  $A$  are disjoint finite sets with  $A_1 \cup A \approx A_2 \cup A$  then  $A_1 \approx A_2$ .

(2) If  $A_1$ ,  $A_2$  and  $A$  are non-empty finite sets with  $A_1 \times A \approx A_2 \times A$  then  $A_1 \approx A_2$ .

*Proof* (1) Suppose  $A_1 \not\approx A_2$ . Then by Theorem 2.4 (and without loss of generality) we can assume that there exists an injective mapping  $h : A_1 \rightarrow A_2$  which is not surjective. This mapping  $h$  can be extended to a mapping  $h' : A_1 \cup A \rightarrow A_2 \cup A$  by putting  $h'(a) = a$  for all  $a \in A$ . Then  $h'$  is injective but not surjective and hence  $A_1 \cup A \not\approx A_2 \cup A$ .

(2) Again suppose  $A_1 \not\approx A_2$  and as in (1) can assume that there exists an injective mapping  $h : A_1 \rightarrow A_2$  which is not surjective. Let  $h' : A_1 \times A \rightarrow A_2 \times A$  be the mapping given by  $h'(a_1, a) = (h(a_1), a)$  for all  $a_1 \in A_1$ ,  $a \in A$ . Then  $h'$  is injective but not surjective and hence  $A_1 \times A \not\approx A_2 \times A$ .  $\square$

**Proposition 2.12** Let  $E$  be an infinite set (i.e.,  $E$  is not finite). Then for each finite set  $A$  there exists a finite subset  $C$  of  $E$  with  $C \approx A$ .

*Proof* Note that if  $C$  is a finite subset of  $E$  then  $C \neq E$  and  $C \cup \{c\}$  is also a finite subset of  $E$  for each  $c \in E \setminus C$ . Let  $A$  be finite set and let  $\mathcal{S}$  denote the set of subsets  $B$  of  $A$  for which there exists a finite subset  $C$  of  $E$  with  $C \approx B$ . Clearly  $\emptyset \in \mathcal{S}$ , thus consider  $B \in \mathcal{S}^p$ , let  $a \in A \setminus B$  and put  $B' = B \cup \{a\}$ . Then there exists a finite subset  $C$  of  $E$  with  $C \approx B$ , since  $B \in \mathcal{S}$ . Also  $C \neq B$  and so choose  $d \in E \setminus C$ . Thus  $C' = C \cup \{d\}$  is a finite subset of  $E$  with  $C' \approx B'$ , which shows that  $B' \in \mathcal{S}$ . Hence  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ .  $\square$

Recall that each set  $E$  the set of non-empty subsets of  $E$  is denoted by  $\mathcal{P}_0(E)$ .

**Proposition 2.13** Let  $A$  be a non-empty finite set and let  $\approx_A$  be the restriction of the equivalence relation  $\approx$  to  $\mathcal{P}_0(A)$ . Let  $\mathcal{E}(A)$  be the set of equivalence classes. Then  $\mathcal{E}(A) \approx A$ .

*Proof* For each  $B \in \mathcal{P}_0(A)$  let  $\approx_B$  be the restriction of  $\approx$  to  $\mathcal{P}_0(B)$  and let  $\mathcal{E}(B)$  be the set of equivalence classes. Also for  $D \in \mathcal{P}_0(B)$  let  $[D]_B$  be the element of  $\mathcal{E}(B)$  containing  $D$ .

Put  $\mathcal{S} = \emptyset \cup \{B \in \mathcal{P}_0(A) : \mathcal{E}(B) \approx B\}$ . Consider  $B \in \mathcal{S}^p$  with  $B \neq \emptyset$ , let  $a \in A \setminus B$  and put  $C = B \cup \{a\}$ . Since  $B \in \mathcal{S}$  there exists a bijective mapping  $\alpha : B \rightarrow \mathcal{E}(B)$  and we extend  $\alpha$  to a mapping  $\alpha' : C \rightarrow \mathcal{E}(C)$  by letting  $\alpha'(a) = [C]_C$  and note

that  $[C]_C$  consists of the single element  $\{C\}$ . Now  $C \not\approx D$  for all  $D \subset B$  and so  $[C]_C \notin \mathcal{E}(B)$ . Thus  $\alpha'$  is injective. But  $\alpha'$  is also surjective: By definition  $\alpha'(a) = [C]_C$  and so consider  $k \in \mathcal{E}(C)$  with  $k \neq [C]_C$  and let  $D \in k$ . Then  $D$  is a proper subset of  $C$  and thus there exists  $D' \subset B$  with  $D' \approx D$ . Since  $\alpha$  is surjective there exists  $d \in B$  with  $\alpha(d) = [D']_B$ . It follows that  $\alpha(d) = [D]_C$ . Hence  $\alpha'$  is bijective which implies that  $C \in \mathcal{S}$ . This shows that  $\mathcal{S}$  is an inductive  $A$ -system and thus  $A \in \mathcal{S}$ , i.e.,  $\mathcal{E}(A) \approx A$ .  $\square$

Some of the results in the following sections involve partial orders and for their proofs Proposition 2.14 below will be needed. Denote by  $\mathbb{B}$  the set of boolean values  $\{\mathsf{T}, \mathsf{F}\}$ .

A *partial order* on a set  $E$  is a mapping  $\leq: E \times E \rightarrow \mathbb{B}$  such that  $e \leq e$  for all  $e \in E$ ,  $e_1 \leq e_2$  and  $e_2 \leq e_1$  both hold if and only if  $e_1 = e_2$ , and  $e_1 \leq e_3$  holds whenever  $e_1 \leq e_2$  and  $e_2 \leq e_3$  for some  $e_2 \in E$ , and where as usual  $e_1 \leq e_2$  is written instead of  $\leq(e_1, e_2) = \mathsf{T}$ .

A *partially ordered set* or *poset* is a pair  $(E, \leq)$  consisting of a set  $E$  and a partial order  $\leq$  on  $E$ .

If  $(E, \leq)$  is a poset and  $D$  is a non-empty subset of  $E$  then  $d \in D$  is said to be a  $\leq$ -*maximal* resp.  $\leq$ -*minimal element* of  $D$  if  $d$  itself is the only element  $e \in D$  with  $d \leq e$  resp. with  $e \leq d$ .

**Proposition 2.14** *If  $(E, \leq)$  is a poset then every non-empty finite subset of  $E$  possesses both a  $\leq$ -maximal and a  $\leq$ -minimal element.*

*Proof* Let  $A$  be a non-empty finite subset of  $E$  and let  $\mathcal{S}$  be the set consisting of the empty set  $\emptyset$  together with those non-empty  $B \in \mathcal{P}(A)$  which possess a  $\leq$ -maximal element. By definition  $\emptyset \in \mathcal{S}$ . Let  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ . We want to show that  $B' = B \cup \{a\} \in \mathcal{S}$ , and this holds trivially if  $B = \emptyset$ , since then  $a$  is the only element in  $B'$ . We can thus suppose that  $B \neq \emptyset$ , in which case  $B$  has a  $\leq$ -maximal element  $b$ . If  $b \leq a$  then  $a$  is a  $\leq$ -maximal element of  $B'$  (since if  $a \leq c$  then  $b \leq c$ , thus  $b = c$  and so  $a = c$ ). On the other hand, if  $b \leq a$  does not hold then  $b$  is still a  $\leq$ -maximal element of  $B'$ . In both cases  $B'$  possesses a  $\leq$ -maximal element and hence  $B \cup \{a\} = B' \in \mathcal{S}$ . This shows  $\mathcal{S}$  is an inductive  $A$ -system and hence  $A \in \mathcal{S}$ , i.e.,  $A$  possesses a  $\leq$ -maximal element. Essentially the same proof also shows that  $A$  possesses a  $\leq$ -minimal element.  $\square$

A partial order  $\leq$  on  $E$  is a *total order* if for all  $e_1, e_2 \in E$  either  $e_1 \leq e_2$  or  $e_2 \leq e_1$  and then  $(E, \leq)$  is called a *totally ordered set*. If  $(E, \leq)$  is a totally ordered set and  $D$  is a non-empty subset of  $E$  then a  $\leq$ -maximal element  $d$  of  $D$  is a  $\leq$ -*maximum*

element, i.e.,  $e \leq d$  for all  $e \in D$ . Moreover, if a  $\leq$ -maximum element exists then it is unique. In the same way, a  $\leq$ -minimal element  $d$  of  $D$  is then a  $\leq$ -*minimum* element, i.e.,  $d \leq e$  for all  $e \in D$ , and if a  $\leq$ -minimum element exists then it is unique.

If  $(E, \leq)$  is a totally ordered set then Proposition 2.14 implies that every non-empty finite subset of  $E$  possesses both a unique  $\leq$ -maximum and a unique  $\leq$ -minimum element.

**Lemma 2.6** *For each finite set  $A$  there exists a totally ordered set  $(E, \leq)$  with  $E \approx A$ .*

*Proof* This is clear: Let  $A$  be a finite set for which there exists a totally ordered set  $(E, \leq)$  with  $E \approx A$  and let  $a \notin A$ . Let  $e$  be an element not in  $E$ . We can extend  $\leq$  to a total order  $\leq'$  on  $E' = E \cup \{e\}$  by defining  $e$  to be the maximum element in  $(E', \leq')$ . Then  $(E', \leq')$  is a totally ordered set with  $E' \approx A \cup \{a\}$ .  $\square$

We end the section with a result which can be used as a replacement for certain kinds of proofs by induction. Let us start by describing the type of situation which is involved here.

Suppose we are working with a set-up in which each finite set comes equipped with some additional structure, so we are dealing with pairs  $(A, \mathcal{T})$ , where  $\mathcal{T}$  is the structure associated with the finite set  $A$ . For example, we might be interested in finite partially ordered sets and in this case  $\mathcal{T}$  would be a partial order defined on  $A$ . It will be the case that for each pair  $(A, \mathcal{T})$  with  $A \neq \emptyset$  and for each  $a \in A$  there is an induced structure  $\mathcal{T}_a$  on  $A \setminus \{a\}$ , resulting in a new pair  $(A \setminus \{a\}, \mathcal{T}_a)$ . Now we would like to show that each such pair  $(A, \mathcal{T})$  has a certain property and a standard approach to tackling this kind of problem is to proceed by induction as follows: For each  $n \in \mathbb{N}$  let  $P(n)$  be the statement that the property holds for all pairs  $(A, \mathcal{T})$  with  $|A| \leq n$  (where  $|A|$  is the cardinality of  $A$ ). It is usually clear that  $P(0)$  holds, thus take  $n \in \mathbb{N} \setminus \{0\}$  and assume  $P(n-1)$  holds. Then in order to verify that  $P(n)$  holds it is enough to show that the property holds for each pair  $(A, \mathcal{T})$  with  $|A| = n$ . Let  $(A, \mathcal{T})$  be such a pair; then for each  $a \in A$  the pair  $(A \setminus \{a\}, \mathcal{T}_a)$  will have the property, since  $|A \setminus \{a\}| < n$ . But the crucial step is to choose the element  $a \in A$  in a way that allows us to deduce that  $(A, \mathcal{T})$  has the property from the fact that  $(A \setminus \{a\}, \mathcal{T}_a)$  does, and the correct choice of  $a$  will clearly depend very much on the structure  $\mathcal{T}$  and the property involved.

Of course, the approach outlined above requires the natural numbers. However, this can be avoided with the help of the result which follows. As an example, it will be applied in Section 11 to give a proof of Dilworth's decomposition theorem.

**Proposition 2.15** *Let  $A$  be a finite set and  $\mathcal{S}$  be a subset of  $\mathcal{P}(A)$  containing  $\emptyset$ . Suppose that each non-empty subset  $F$  of  $A$  contains an element  $s_F$  such that  $F \in \mathcal{S}$  whenever  $\mathcal{P}(F \setminus \{s_F\}) \subset \mathcal{S}$ . Then  $\mathcal{S} = \mathcal{P}(A)$ .*

*Proof* Let  $\mathcal{S}_*$  consist of those  $B \in \mathcal{P}(A)$  for which  $\{E \in \mathcal{P}(A) : E \preceq B\} \subset \mathcal{S}$ . Thus  $B \in \mathcal{S}_*$  if and only if  $E \in \mathcal{S}$  for each  $E \subset A$  with  $E \preceq B$ . We will show that  $\mathcal{S}_*$  is an inductive  $A$ -system. It then follows that  $\mathcal{S}_* = \mathcal{P}(A)$  and hence that  $\mathcal{S} = \mathcal{P}(A)$ , since  $\mathcal{S}_* \subset \mathcal{S}$ .

To start with it is clear that  $\emptyset \in \mathcal{S}_*$ , since  $E \preceq \emptyset$  is only possible with  $E = \emptyset$  and  $\emptyset \in \mathcal{S}$ . Thus let  $B \in \mathcal{S}_*$  and  $a \in A \setminus B$ ; we want to show that  $B \cup \{a\} \in \mathcal{S}_*$  i.e., to show that if  $F \in \mathcal{P}(A)$  with  $F \preceq B \cup \{a\}$  then  $F \in \mathcal{S}$ . If  $F \not\approx B \cup \{a\}$  then  $F \preceq B$ , and in this case  $F \in \mathcal{S}$ , since  $B \in \mathcal{S}_*$ . On the other hand, if  $F \approx B \cup \{a\}$  then  $F \setminus \{s_F\} \approx B$  and  $B \in \mathcal{S}_*$ , and so in particular  $\mathcal{P}(F \setminus \{s_F\}) \subset \mathcal{S}$ . Hence also  $F \in \mathcal{S}$ , i.e.,  $\mathcal{S}$  contains every subset  $F$  of  $A$  with  $F \preceq B \cup \{a\}$ . Therefore  $B \cup \{a\} \in \mathcal{S}_*$ , which shows that  $\mathcal{S}_*$  is an inductive  $A$ -system.  $\square$

### 3 Iterators and assignments

As mentioned in the Introduction we will be dealing with mappings defined on the collection of all finite sets. This collection is too large to be considered a set, meaning that treating it as a set could lead to various paradoxes. Such a large collection is called a *class* and in particular the class of all finite sets will be denoted by  $\mathbf{Fin}$ . However, as far as what will be found in these notes, there is no problem in treating classes as if they were just sets.

In this section we introduce what will be called an assignment of finite sets in a triple  $\mathbb{I} = (X, f, x_0)$ , where  $X$  is some class of objects,  $f : X \rightarrow X$  is a mapping of the class  $X$  into itself and  $x_0$  is an object of  $X$ . Such a triple will be called an *iterator*. The results of this section will be applied in Section 4 to define the finite ordinals. In this case  $(X, f, x_0) = (\mathbf{Fin}, \sigma, \emptyset)$ , where  $\sigma : \mathbf{Fin} \rightarrow \mathbf{Fin}$  is the mapping given by  $\sigma(A) = A \cup \{A\}$  for each finite set  $A$ .

The archetypal example of an iterator whose first component is a set is  $(\mathbb{N}, \mathbf{s}, 0)$ , where the successor mapping  $\mathbf{s} : \mathbb{N} \rightarrow \mathbb{N}$  is given by  $\mathbf{s}(n) = n + 1$  for each  $n \in \mathbb{N}$ . However, we will also be dealing with examples in which the first component is a finite set. In what follows let us fix an iterator  $\mathbb{I} = (X, f, x_0)$ .

A mapping  $\omega : \mathbf{Fin} \rightarrow X$  will be called an *assignment of finite sets in  $\mathbb{I}$*  or, when it is clear what  $\mathbb{I}$  is, simply an *assignment of finite sets* if  $\omega(\emptyset) = x_0$  and

$$\omega(A \cup \{a\}) = f(\omega(A))$$

for each finite set  $A$  and each element  $a \notin A$ .

For each finite set  $A$  denote the cardinality of  $A$  by  $|A|$  (with  $|A|$  defined as usual in terms of  $\mathbb{N}$ ). Then it is clear that the mapping  $|\cdot| : \mathbf{Fin} \rightarrow \mathbb{N}$  defines an assignment of finite sets in  $(\mathbb{N}, \mathbf{s}, 0)$  and that this is the unique such assignment.

Let  $X$  be a class and let  $\lambda : \mathbf{Fin} \rightarrow X$  be a mapping. For each  $A \in \mathbf{Fin}$  let  $\lambda_A : \mathcal{P}(A) \rightarrow X$  be the restriction of  $\lambda$  to  $\mathcal{P}(A)$ . Then the family of mappings  $\{\lambda_A : A \in \mathbf{Fin}\}$ , is *compatible* in the sense that whenever  $A, B \in \mathbf{Fin}$  with  $B \subset A$  then  $\lambda_B$  is the restriction of  $\lambda_A$  to  $\mathcal{P}(B)$ , i.e.,  $\lambda_B(C) = \lambda_A(C)$  for all  $C \in \mathcal{P}(B)$ .

**Lemma 3.1** *Let  $\{\lambda_A : A \in \mathbf{Fin}\}$ , be a compatible family and define a mapping  $\lambda : \mathbf{Fin} \rightarrow X$  by setting  $\lambda(A) = \lambda_A(A)$  for each  $A \in \mathbf{Fin}$ . Then  $\lambda_A$  is the restriction of  $\lambda$  to  $\mathcal{P}(A)$  for each  $A \in \mathbf{Fin}$ .*

*Proof* This is clear, since if  $B \in \mathcal{P}(A)$  then  $\lambda(B) = \lambda_B(B) = \lambda_A(B)$ .  $\square$

In general, we assume that the following statements are valid for mappings between classes:



- (1) Mappings are determined by their values. This implies in particular that the mapping  $\lambda : \mathbf{Fin} \rightarrow X$  defined in terms of a compatible family, is unique.
- (2) For each class  $X$  there is a mapping  $\text{id}_X : X \rightarrow X$  satisfying  $\text{id}_X(x) = x$  for all  $x \in X$ .
- (3) If  $X, Y$  and  $Z$  are classes and  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are mappings then there is a mapping  $g \circ f : X \rightarrow Z$  (their composition) satisfying  $(g \circ f)(x) = g(f(x))$  for all  $x \in X$ .

We will also need the following which corresponds to Proposition 2.7:

**Proposition 3.1** *Let  $X, Y$  and  $Z$  be classes, let  $f : Z \rightarrow X$  be a surjective mapping and let  $g : Z \rightarrow Y$  be a mapping. Then there exists a mapping  $h : X \rightarrow Y$  with  $g = h \circ f$  if and only if  $g(z) = g(z')$  whenever  $z, z' \in Z$  with  $f(z) = f(z')$ . Moreover, if  $h$  exists then it is unique.*

*Proof* If we assume that there is a one-to-one correspondence between mappings and graphs then the proof of Proposition 2.7 can be used. Without this assumption we can proceed as follows: Suppose first that there exists  $f : Z \rightarrow X$  with  $g = h \circ f$ . If  $z, z' \in Z$  with  $f(z) = f(z')$  then  $g(z) = h(f(z)) = h(f(z')) = g(z')$  and so  $g(z) = g(z')$  whenever  $z, z' \in Z$  with  $f(z) = f(z')$ . Moreover,  $h(f(z)) = g(z)$  for each  $z \in Z$  and  $f$  is surjective and hence  $h$  is uniquely determined by  $f$  and  $g$ .

Suppose conversely that  $g(z) = g(z')$  whenever  $z, z' \in Z$  with  $f(z) = f(z')$ . For each  $x \in X$  let  $G_x = \{z \in Z : f(z) = x\}$ . Thus  $G_x \neq \emptyset$ , since  $f$  is surjective and if  $x \neq x'$  then  $G_x \cap G_{x'} = \emptyset$ . Let  $\mathcal{E} = \{E \in \mathcal{P}(Z) : E = G_x \text{ for some } x \in X\}$  and define  $r : X \rightarrow \mathcal{E}$  by  $r(x) = G_x$  for each  $x \in X$ . Hence  $r$  is a bijection. Now if  $z, z' \in r(x)$  then  $f(z) = f(z')$  and so  $g(z) = g(z')$ . There is thus a mapping  $q : \mathcal{E} \rightarrow Y$  such that  $q(r(x)) = g(z)$ , where  $z$  is any element in  $r(x)$  and note that  $g(z)$  does not depend on which element of  $r(x)$  is used. Define  $h : X \rightarrow Y$  by  $h = q \circ r$  and so  $h \circ f = q \circ r \circ f$ . Let  $z \in Z$ ; then  $x = f(z) \in X$  and thus  $r(x) = G_x \in \mathcal{E}$ . Hence  $q(G_x) = q(r(x)) = g(z)$ , since  $z \in r(x)$ , i.e.,  $(h \circ f)(z) = g(z)$ , which shows that  $h \circ f = g$ .  $\square$

**Proposition 3.2** *Let  $X$  and  $Y$  be classes and let  $f : Y \rightarrow X$  be a bijection. Then there exists the inverse mapping  $f^{-1} : X \rightarrow Y$ . This is the unique mapping  $g : X \rightarrow Y$  satisfying  $g \circ f = \text{id}_Y$  and  $f \circ g = \text{id}_X$ .*

*Proof* If we again assume that there is a one-to-one correspondence between mappings and graphs then the proof of Proposition 2.8 based on Proposition 2.6 can be used. If not then we can proceed as follows: Let  $f : Y \rightarrow X$  be a bijection.

Set  $Z = Y$  and so  $f : Z \rightarrow X$  is a bijection. Also put  $p : Z \rightarrow Y = \text{id}_Y$ . If  $z, z' \in Z$  with  $f(z) = f(z')$  then  $z = z'$ , since  $f$  is a bijection and so  $p(z) = p(z')$ . Thus by Proposition 3.1 there exists a unique mapping  $g : X \rightarrow Y$  such that  $p = g \circ f$ , i.e., with  $g \circ f = \text{id}_Y$ . Repeating the above construction with the bijection  $g : X \rightarrow Y$  there exists a bijection  $f' : Y \rightarrow X$  such that  $f' \circ g = \text{id}_X$ . Then  $f' = f \circ g \circ f' = f$ , i.e.,  $f' = f$ . This shows that  $g = f^{-1}$ .  $\square$

**Theorem 3.1** (1) *There exists a unique assignment  $\omega$  of finite sets in  $\mathbb{I}$ .*

(2) *If  $A$  and  $B$  are finite sets with  $A \approx B$  then  $\omega(A) = \omega(B)$ .*

*Proof* Let  $A$  be a finite set; then a mapping  $\omega_A : \mathcal{P}(A) \rightarrow X$  will be called an  $A$ -assignment if  $\omega_A(\emptyset) = x_0$  and  $\omega_A(B \cup \{a\}) = f(\omega_A(B))$  for each proper subset  $B$  of  $A$  and each  $a \in A \setminus B$ .

**Lemma 3.2** *For each finite set  $A$  there exists a unique  $A$ -assignment.*

*Proof* Let  $A$  be a finite set and let  $\mathcal{S}$  be the set consisting of those  $B \in \mathcal{P}(A)$  for which there exists a unique  $B$ -assignment. Then  $\emptyset \in \mathcal{S}$ , since the mapping  $\omega_\emptyset : \mathcal{P}(\emptyset) \rightarrow X$  with  $\omega_\emptyset(\emptyset) = x_0$  is clearly the unique  $\emptyset$ -assignment.

Let  $B \in \mathcal{S}^p$  with unique  $B$ -assignment  $\omega_B$ , and let  $a \in A \setminus B$ ; put  $B' = B \cup \{a\}$ . Now  $\mathcal{P}(B')$  is the disjoint union of the sets  $\mathcal{P}(B)$  and  $\{C \cup \{a\} : C \subset B\}$  and so we can define a mapping  $\omega_{B'} : \mathcal{P}(B') \rightarrow X$  by letting  $\omega_{B'}(C) = \omega_B(C)$  and  $\omega_{B'}(C \cup \{a\}) = f(\omega_B(C))$  for each  $C \subset B$ . Then  $\omega_{B'}(\emptyset) = \omega_B(\emptyset) = x_0$ , and so consider  $C' \subset B'$  and  $b \in B' \setminus C'$ . There are three cases:

The first is with  $C' \subset B$  and  $b \in B \setminus C'$  and here

$$\omega_{B'}(C' \cup \{b\}) = \omega_B(C' \cup \{b\}) = f(\omega_B(C')) = f(\omega_{B'}(C')) .$$

The second is with  $C' \subset B$  and  $b = a$ . In this case

$$\omega_{B'}(C' \cup \{b\}) = \omega_{B'}(C' \cup \{a\}) = f(\omega_B(C')) = f(\omega_{B'}(C')) .$$

The final case is with  $C' = C \cup \{a\}$  for some  $C \subset B$  and  $b \in B \setminus C$ , and here

$$\begin{aligned} \omega_{B'}(C' \cup \{b\}) &= \omega_{B'}(C \cup \{a\} \cup \{b\}) = f(\omega_B(C \cup \{b\})) \\ &= f(f(\omega_B(C))) = f(\omega_{B'}(C \cup \{a\})) = f(\omega_{B'}(C')) . \end{aligned}$$

In all three cases  $\omega_{B'}(C' \cup \{b\}) = f(\omega_{B'}(C'))$ , which shows  $\omega_{B'}$  is a  $B'$ -assignment.

Now let  $\omega'_{B'}$  be an arbitrary  $B'$ -assignment. In particular  $\omega'_{B'}(C \cup \{b\}) = f(\omega'_{B'}(C))$  for all  $C \subset B$  and all  $b \in B \setminus C$ , and from the uniqueness of the  $B$ -assignment  $\omega_B$  it follows that  $\omega'_{B'}(C) = \omega_B(C)$  and thus also that

$$\omega'_{B'}(C \cup \{a\}) = f(\omega'_{B'}(C)) = f(\omega_B(C)) = \omega_B(C \cup \{a\})$$

for all  $C \subset B$ , i.e.,  $\omega'_{B'} = \omega_B$ . Hence  $B \cup \{a\} \in \mathcal{S}$ .

Therefore  $\mathcal{S}$  is an inductive  $A$ -system and thus  $A \in \mathcal{S}$ . This shows there exists a unique  $A$ -assignment.  $\square$

**Lemma 3.3** *If  $A, B \in \mathbf{Fin}$  with  $B \subset A$ ; then the unique  $B$ -assignment  $\omega_B$  is the restriction of the unique  $A$ -assignment  $\omega_A$  to  $\mathcal{P}(B)$ .*

*Proof* This follows immediately from the uniqueness of  $\omega_B$ .  $\square$

Lemma 3.3 shows that the family  $\{\omega_A : A \in \mathbf{Fin}\}$  is compatible and therefore there exists a unique mapping  $\omega : \mathbf{Fin} \rightarrow X$  such that  $\omega_A$  is the restriction of  $\omega$  to  $\mathcal{P}(A)$  for each  $A \in \mathbf{Fin}$ . In particular,  $\omega(A) = \omega_A(A)$  for each  $A \in \mathbf{Fin}$ . Thus  $\omega(\emptyset) = \omega_\emptyset(\emptyset) = x_0$  and if  $A \in \mathbf{Fin}$  and  $a \notin A$  then by Lemma 3.3

$$\omega(A \cup \{a\}) = \omega_{A \cup \{a\}}(A \cup \{a\}) = f(\omega_{A \cup \{a\}}(A)) = f(\omega_A(A)) = f(\omega(A)).$$

Hence  $\omega$  is an assignment of finite sets in  $\mathbb{I}$ . For the uniqueness consider an arbitrary assignment  $\omega'$  of finite sets in  $\mathbb{I}$ . Then for each  $A \in \mathbf{Fin}$  the restriction of  $\omega'$  to  $\mathcal{P}(A)$  is an  $A$ -iterator and thus equal to  $\omega_A$ . It follows that  $\omega' = \omega$ . This shows that there is a unique assignment  $\omega$  of finite sets in  $\mathbb{I}$ .

(2) We must show that if  $A$  and  $B$  are finite sets with  $A \approx B$  then  $\omega(A) = \omega(B)$ . Let  $A$  be a finite set and  $\mathcal{S}$  be the set consisting of those  $C \in \mathcal{P}(A)$  for which  $\omega(C) = \omega(B)$  whenever  $B$  is a finite set with  $B \approx C$ . Then  $\emptyset \in \mathcal{S}$ , since  $B \approx \emptyset$  if and only if  $B = \emptyset$ . Consider  $C \in \mathcal{S}^p$  and  $a \in A \setminus C$ , and let  $B$  be a finite set with  $B \approx C \cup \{a\}$ ; thus  $B \neq \emptyset$ , so let  $b \in B$ . Then  $B' = B \setminus \{b\} \approx C$  and hence  $\omega(B') = \omega(C)$ . Thus  $\omega(B) = \omega(B' \cup \{b\}) = f(\omega(B')) = f(\omega(C)) = \omega(C \cup \{a\})$ . This shows that  $C \cup \{a\} \in \mathcal{S}$ . Therefore  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ , i.e.,  $\omega(A) = \omega(B)$  whenever  $A \approx B$ . This completes the proof of Theorem 3.1.  $\square$

For what follows it is necessary to determine the range of the assignment  $\omega$ , this being the subclass  $X_0 = \{x \in X : x = \omega(A) \text{ for some finite set } A\}$  of  $X$ . A subclass  $Y$  of  $X$  is said to be *f-invariant* if  $f(y) \in Y$  for all  $y \in Y$ . The next result shows that  $X_0$  is the least *f-invariant* subclass of  $X$  containing  $x_0$ .

**Lemma 3.4**  *$X_0$  is an  $f$ -invariant subclass of  $X$  containing  $x_0$ . Moreover, if  $X'$  is any  $f$ -invariant subclass of  $X$  containing  $x_0$  then  $X_0 \subset X'$ .*

*Proof* Clearly  $x_0 \in X_0$  since  $x_0 = \omega(\emptyset)$ . Thus let  $x \in X_0$ , and so there exists a finite set  $A$  with  $x = \omega(A)$ . By Lemma 1.3 there exists an element not in  $A$ ; it then follows that  $\omega(A \cup \{a\}) = f(\omega(A)) = f(x)$ , which implies that  $f(x) \in X_0$ . Hence  $X_0$  is  $f$ -invariant. Now let  $X'$  be any  $f$ -invariant subclass of  $X$  containing  $x_0$ . Let  $A$  be a finite set and let  $\mathcal{S} = \{B \in \mathcal{P}(A) : \omega(B) \in X'\}$ . Then  $\emptyset \in \mathcal{S}$  since  $\omega(\emptyset) = x_0 \in X'$ . Consider  $B \in \mathcal{S}^p$  (and so  $\omega(B) \in X'$ ) and let  $a \in A \setminus B$ . Then  $\omega(B \cup \{a\}) = f(\omega(B)) \in X'$ , since  $X'$  is  $f$ -invariant, and hence  $B \cup \{a\} \in \mathcal{S}$ . Thus  $\mathcal{S}$  is an inductive  $A$ -system and hence  $A \in \mathcal{S}$ , i.e.,  $\omega(A) \in X'$ . This shows that  $\omega(A) \in X'$  for each finite set  $A$  and it follows that  $X_0 \subset X'$ .  $\square$

The iterator  $\mathbb{I}$  is said to be *minimal* if the only  $f$ -invariant subclass of  $X$  containing  $x_0$  is  $X$  itself, thus  $\mathbb{I}$  is minimal if and only if  $X_0 = X$ . In particular, it is easy to see that the Principle of Mathematical Induction is exactly the requirement that the iterator  $(\mathbb{N}, s, 0)$  be minimal.

Note that the iterator  $\mathbb{I}_0 = (X_0, f_0, x_0)$  is always minimal, where  $f_0 : X_0 \rightarrow X_0$  is the restriction of  $f$  to  $X_0$ .

For a minimal iterator Lemma 3.4 takes the form:

**Lemma 3.5** *If  $\mathbb{I}$  is minimal then for each  $x \in X$  there exists a finite set  $A$  with  $x = \omega(A)$ . Thus the mapping  $\omega : \text{Fin} \rightarrow X$  is surjective.*

*Proof* This is a special case of Lemma 3.4.  $\square$

From now on we will make use of Lemma 1.3 (guaranteeing the existence of an element  $a$  not in a set  $A$ ) without referring explicitly to this result.

**Proposition 3.3** *Suppose  $\mathbb{I}$  is minimal; then  $\{x_0\} \cup f(X) = X$ . Thus for each  $x \neq x_0$  there exists an  $x' \in X$  with  $x = f(x')$ . Moreover, the mapping  $f$  is surjective if and only if  $x_0 \in f(X)$ .*

*Proof* For a general iterator the subclass  $(\{x_0\} \cup f(X))$  is always  $f$ -invariant and contains  $x_0$ . Thus, since  $\mathbb{I}$  is minimal it follows that  $\{x_0\} \cup f(X) = X$ .  $\square$

The iterator  $\mathbb{I}$  will be called *proper* if  $B_1 \approx B_2$  whenever  $B_1$  and  $B_2$  are finite sets with  $\omega(B_1) = \omega(B_2)$ . If  $\mathbb{I}$  is proper then by Theorem 3.1 (2)  $B_1 \approx B_2$  holds if and only if  $\omega(B_1) = \omega(B_2)$ .

Note that if  $\mathbb{I}$  is proper then so is the minimal iterator  $\mathbb{I}_0 = (X_0, f_0, x_0)$ .

$\mathbb{I}$  will be called a *Peano iterator* if it is minimal and  $\mathbb{N}$ -like, where  $\mathbb{N}$ -like means that the mapping  $f$  is injective and  $x_0 \notin f(X)$ . The Peano axioms thus require  $(\mathbb{N}, s, 0)$  to be a Peano iterator. If  $\mathbb{I}$  is a Peano iterator then  $f$  is injective and so Proposition 3.3 implies that for each  $x \neq x_0$  there exists a unique  $x' \in X$  with  $x = f(x')$ .

Theorem 3.2 below states that a minimal iterator is proper if and only if it is a Peano iterator. This will be applied to prove the recursion theorem for Peano iterators.

**Theorem 3.2** *A minimal iterator  $\mathbb{I}$  is proper if and only if it is  $\mathbb{N}$ -like, i.e., if and only if it is a Peano iterator.*

*Proof* Assume first that  $\mathbb{I}$  is  $\mathbb{N}$ -like. Let  $A$  be a finite set and let  $\mathcal{S}$  denote the set of subsets  $B$  of  $A$  such that  $B \approx B'$  whenever  $B' \subset A$  with  $\omega(B') = \omega(B)$ . Let  $B \subset A$  with  $B \neq \emptyset$ , let  $b \in B$  and put  $B' = B \setminus \{b\}$ . It then follows that  $\omega(B) = \omega(B' \cup \{b\}) = f(\omega(B'))$ , and so  $\omega(B) \neq x_0$ , since  $x_0 \notin f(X)$ . Thus  $\omega(B) \neq \omega(\emptyset)$ , which shows that  $\emptyset \in \mathcal{S}$ , since  $\emptyset \approx B$  if and only if  $B = \emptyset$ .

Let  $B \in \mathcal{S}^p$  and let  $a \in A \setminus B$ . Consider  $B' \subset A$  with  $\omega(B') = \omega(B \cup \{a\})$ ; then  $\omega(B') = f(\omega(B)) \in f(X)$ , hence  $\omega(B') \neq x_0$  and so  $B' \neq \emptyset$ . Let  $b \in B'$  and put  $C = B' \setminus \{b\}$ ; then  $f(\omega(C)) = \omega(C \cup \{b\}) = \omega(B') = f(\omega(B))$  and thus  $\omega(C) = \omega(B)$ , since  $f$  is injective, and it follows that  $C \approx B$ , since  $B \in \mathcal{S}$ . But  $B' = C \cup \{b\}$  with  $b \notin C$ ,  $a \notin B$  and  $C \approx B$ , and therefore  $B' = C \cup \{b\} \approx B \cup \{a\}$ . Hence  $B \cup \{a\} \in \mathcal{S}$ , which shows that  $\mathcal{S}$  is an inductive  $A$ -system and thus that  $\mathcal{S} = \mathcal{P}(A)$ . This implies that if  $B_1, B_2$  are subsets of  $A$  with  $\omega(B_1) = \omega(B_2)$  then  $B_1 \approx B_2$ .

Now let  $B_1$  and  $B_2$  be arbitrary finite sets with  $\omega(B_1) = \omega(B_2)$ . Applying the above with  $A = B_1 \cup B_2$  then shows that  $B_1 \approx B_2$ . Thus  $\mathbb{I}$  is proper.

For the converse we assume  $\mathbb{I}$  is not  $\mathbb{N}$ -like and show this implies it is not proper. Suppose first that  $x_0 = f(x)$  for some  $x \in X$ . By Lemma 3.5 there exists a finite set  $A$  with  $x = \omega(A)$  and there exists some element  $a$  not in  $A$ . Then  $A \cup \{a\} \not\approx \emptyset$  but  $\omega(A \cup \{a\}) = f(\omega(A)) = f(x) = x_0 = \omega(\emptyset)$ . Thus  $(X, f, x_0)$  is not proper. Suppose now that  $f$  is not injective and so there exist  $x, x' \in X$  with  $x \neq x'$  and  $f(x) = f(x')$ . By Lemma 3.5 there exist finite sets  $A$  and  $B$  with  $x = \omega(A)$  and  $x' = \omega(B)$  and by Theorem 2.4 and Lemma 2.1 we can assume that  $B \subset A$ . Thus  $B$  is a proper subset of  $A$ , since  $\omega(A) = x \neq x' = \omega(B)$ . Let  $a \notin A$ ; then  $B \cup \{a\}$  is a proper subset of  $A \cup \{a\}$  and so by Theorem 2.2  $B \cup \{a\} \not\approx A \cup \{a\}$ . But  $\omega(B \cup \{a\}) = f(\omega(B)) = f(x') = f(x) = f(\omega(A)) = \omega(A \cup \{a\})$ , which again shows  $\mathbb{I}$  is not proper.  $\square$

Here is the recursion theorem (which first appeared in Dedekind [1]).

**Theorem 3.3** *If  $\mathbb{I}$  is a Peano iterator then for each iterator  $\mathbb{J} = (Y, g, y_0)$  there exists a unique mapping  $\pi : X \rightarrow Y$  with  $\pi(x_0) = y_0$  such that  $\pi \circ f = g \circ \pi$ .*

*Proof* As before let  $\omega$  be the assignment of finite sets in  $\mathbb{I}$  and denote the assignment of finite sets in  $\mathbb{J}$  by  $\omega'$ . If  $A, B \in \mathbf{Fin}$  with  $\omega(A) = \omega(B)$  then by Theorem 3.2  $A \approx B$  and therefore by Theorem 3.1 (2)  $\omega'(A) = \omega'(B)$ . Moreover, by Lemma 3.5  $\omega$  is surjective and thus by Proposition 3.1 there exists a unique factor mapping  $\pi : X \rightarrow Y$  such that  $\pi(\omega(A)) = \omega'(A)$  for each  $A \in \mathbf{Fin}$ . In particular,  $\pi(x_0) = \pi(\omega(\emptyset)) = \omega'(\emptyset) = y_0$ . Let  $x \in X$ ; as above there exists a finite set  $A$  with  $x = \omega(A)$ , and there exists an element  $a$  not contained in  $A$ . Hence

$$\begin{aligned} \pi(f(x)) &= \pi(f(\omega(A))) = \pi(\omega(A \cup \{a\})) \\ &= \omega'(A \cup \{a\}) = g(\omega'(A)) = g(\pi(\omega(A))) = g(\pi(x)) \end{aligned}$$

and this shows that  $\pi \circ f = g \circ \pi$ .

The proof of the uniqueness only uses the fact that  $\mathbb{I}$  is minimal: Let  $\pi' : X \rightarrow Y$  be a further mapping with  $\pi'(x_0) = y_0$  and such that  $\pi' \circ f = g \circ \pi'$  and let  $X' = \{x \in X : \pi(x) = \pi'(x)\}$ . Then  $x_0 \in X'$ , since  $\pi(x_0) = y_0 = \pi'(x_0)$ , and if  $x \in X'$  then  $\pi'(f(x)) = g(\pi'(x)) = g(\pi(x)) = \pi(f(x))$ , i.e.,  $f(x) \in X'$ . Thus  $X'$  is an  $f$ -invariant subclass of  $X$  containing  $x_0$  and so  $X' = X$ , i.e.,  $\pi' = \pi$ .  $\square$

The Peano axioms require  $(\mathbb{N}, s, 0)$  to be a Peano iterator and hence for each iterator  $\mathbb{J} = (Y, g, y_0)$  there exists a unique mapping  $\pi : \mathbb{N} \rightarrow Y$  with  $\pi(0) = y_0$  such that  $\pi \circ s = g \circ \pi$ .

**Theorem 3.4** *Let  $\mathbb{I}$  be minimal; then the class  $X$  is a finite set if and only if  $\mathbb{I}$  is not proper.*

*Proof* Suppose first that  $X$  is a finite set. Since  $\mathbb{I}$  is minimal Proposition 3.3 states that  $f$  is surjective if and only if  $x_0 \in f(X)$ , and since  $X$  is a finite set Theorem 2.1 implies  $f$  is surjective if and only if it is injective. Therefore either  $x_0 \in f(X)$  or  $f$  is not injective, which means that  $\mathbb{I}$  is not  $\mathbb{N}$ -like. It thus follows from Theorem 3.2 that  $\mathbb{I}$  is not proper. This can also be shown directly without using Theorem 3.2: Assume first that  $x_0 = f(x)$  for some  $x \in X$ . By Lemma 3.5 there exists a finite set  $A$  with  $x = \omega(A)$ ; let  $a$  be some element not in  $A$ . Then  $A \cup \{a\} \not\approx \emptyset$  but  $\omega(A \cup \{a\}) = f(\omega(A)) = f(x) = x_0 = \omega(\emptyset)$ . Thus  $\mathbb{I}$  is not proper.

Assume now that  $f$  is not injective and so there exist  $x, x' \in X$  with  $x \neq x'$  and  $f(x) = f(x')$ . By Lemma 3.5 there exist finite sets  $A$  and  $B$  with  $x = \omega(A)$  and  $x' = \omega(B)$  and by Theorem 2.4 and Lemma 2.1 we can assume that  $B \subset A$ . Thus  $B$  is a proper subset of  $A$ , since  $\omega(A) = x \neq x' = \omega(B)$ . Let  $a \notin A$ ; then  $B \cup \{a\}$  is a proper subset of  $A \cup \{a\}$  and so by Theorem 2.2  $B \cup \{a\} \not\approx A \cup \{a\}$ . But  $\omega(B \cup \{a\}) = f(\omega(B)) = f(x') = f(x) = f(\omega(A)) = \omega(A \cup \{a\})$ , which again shows  $(X, f, x_0)$  is not proper.

Suppose conversely that  $\mathbb{I}$  is not proper, so there exist finite sets  $A$  and  $A'$  with  $\omega(A) = \omega(A')$  and  $A \not\approx A'$ . Then by Lemma 2.2 and Theorem 3.3 there exist such subsets  $A$  and  $A'$  with  $A'$  a proper subset of  $A$ . We show that for each finite set  $B$  there exists  $C \subset A$  with  $\omega(C) = \omega(B)$ . By Lemma 3.5 it then follows that the mapping  $\omega_A : \mathcal{P}(A) \rightarrow X$  with  $\omega_A(B) = \omega(B)$  for each  $B \subset A$  is surjective, and hence by Proposition 2.2 (2) that  $X$  is a finite set, since by Proposition 2.3  $\mathcal{P}(A)$  is finite.

Thus let  $B$  be a finite set; by Theorem 2.4 and Lemma 2.1 there exists a finite set  $D$  with  $D \approx B$  and either  $D \subset A$  or  $A \subset D$ , and by Theorem 3.1 (2)  $\omega(D) = \omega(B)$ . If  $D \subset A$  then  $C = D$  is the required subset of  $A$ . It remains to show that if  $D$  is a finite set with  $A \subset D$  then there exists  $C \subset A$  with  $\omega(C) = \omega(D)$ .

Thus let  $D$  be a finite set with  $A \subset D$ . Put  $D' = D \setminus A$  and let  $\mathcal{S}$  be the set consisting of those  $E \in \mathcal{P}(D')$  for which there exists  $C \subset A$  with  $\omega(C) = \omega(A \cup E)$ . In particular  $\emptyset \in \mathcal{S}$ . Consider  $E \in \mathcal{S}^p$  and so  $\omega(C) = \omega(A \cup E)$  for some  $C \subset A$ , let  $b \in D' \setminus E$ . If  $C$  is a proper subset of  $A$  and  $a \in A \setminus C$  then  $C \cup \{a\} \subset A$  and  $\omega(C \cup \{a\}) = f(\omega(C)) = f(\omega(A \cup E)) = \omega(A \cup (E \cup \{b\}))$ . On the other hand, if  $C = A$  and  $a \in A \setminus A'$  then  $A' \cup \{a\} \subset A$  and

$$\omega(A \cup (E \cup \{b\})) = f(\omega(A \cup E)) = f(\omega(C)) = f(\omega(A)) = f(\omega(A')) = \omega(A' \cup \{a\}) .$$

Thus  $E \cup \{b\} \in \mathcal{S}$ , which shows  $\mathcal{S}$  is an inductive  $D'$ -system. Therefore  $D' \in \mathcal{S}$  and hence there exists  $C \subset A$  with  $\omega(C) = \omega(A \cup D') = \omega(D)$ .  $\square$

Note that the proof of Proposition 2.2 (2) is still valid in the form we have used here: If there is a surjective mapping  $h : A \rightarrow X$  with  $A$  a finite set and  $X$  a class then  $X$  is a finite set.

Theorems 3.2 and 3.4 imply that for a minimal iterator  $\mathbb{I}$  there are two mutually exclusive possibilities: Either  $\mathbb{I}$  is a Peano iterator or  $X$  is a finite set.

**Proposition 3.4** *Let  $\mathbb{I}$  be minimal. If  $x_0 \in f(X)$  then  $X$  is a finite set and the mapping  $f$  is bijective.*

*Proof* Exactly as in the proof above the fact that  $x_0 \in f(X)$  implies  $\mathbb{I}$  is not proper, and thus by Theorem 3.4  $X$  is a finite set. Moreover, by Proposition 3.3  $f$  is surjective, since  $x_0 \in f(X)$ , and therefore by Theorem 2.1  $f$  is bijective, since  $X$  is a finite set.  $\square$

The recursion theorem will now be applied to give another proof of the fact that the definition of a finite set being used here is equivalent to the usual one. The usual definition of  $A$  being finite is that there exists  $n \in \mathbb{N}$  and a bijective mapping  $h : [n] \rightarrow A$ , where  $[0] = \emptyset$  and  $[n] = \{0, 1, \dots, n-1\}$  for  $n \in \mathbb{N} \setminus \{0\}$ . Moreover, if there is a bijective mapping  $h : [n] \rightarrow A$ , then  $n$  is the cardinality of  $A$ , i.e.,  $n = |A|$ , and so  $A \approx [|A|]$  for each finite set  $A$ . The problem here is to assign a meaning to the expression  $\{0, 1, \dots, n-1\}$ , and one way to do this is to make use of the fact that  $\{0, 1, \dots, n\} = \{0, 1, \dots, n-1\} \cup \{n\}$  and hence that  $[s(n)] = [n] \cup \{n\}$  for all  $n \in \mathbb{N}$ . The corresponding approach works with any Peano iterator  $\mathbb{I}$ .

We start with a general construction. For each class  $X$  let  $\text{Fin}(X)$  denote the class of all finite subsets of  $X$ .

Let  $\mathbb{I} = (X, f, x_0)$  be an iterator and consider the iterator  $\mathbb{I}^{\text{Fin}} = (\text{Fin}(X), f^{\text{Fin}}, \emptyset)$ , where  $f^{\text{Fin}} : \text{Fin}(X) \rightarrow \text{Fin}(X)$  is defined by letting  $f^{\text{Fin}}(A) = \{x_0\} \cup f(A)$  for all  $A \in \text{Fin}(X)$ . Let  $\omega : \text{Fin} \rightarrow X$  be the assignment of finite sets in  $\mathbb{I}$  and  $\Omega : \text{Fin} \rightarrow \text{Fin}(X)$  be the assignment of finite sets in  $\mathbb{I}^{\text{Fin}}$ . Moreover, let  $\mathbb{I}_0^{\text{Fin}} = (\text{Fin}(X)_0, f_0^{\text{Fin}}, \emptyset)$  be the corresponding minimal iterator, so  $\text{Fin}(X)_0$  is the least  $f^{\text{Fin}}$ -invariant subclass of  $\text{Fin}(X)$  containing  $\emptyset$  and  $f_0^{\text{Fin}} : \text{Fin}(X)_0 \rightarrow \text{Fin}(X)_0$  is the restriction of  $f^{\text{Fin}}$  to  $\text{Fin}(X)_0$ . Therefore by Lemma 3.4

$$\text{Fin}(X)_0 = \{U \in \text{Fin}(X) : U = \Omega(A) \text{ for some finite set } A\}.$$

**Theorem 3.5** *Suppose that  $\mathbb{I}$  is a Peano iterator. Then:*

- (1)  $\Omega(A) \approx A$  for each finite set  $A$ . Thus if  $A$  and  $B$  are finite sets then  $A \approx B$  if and only if  $\Omega(A) = \Omega(B)$ . Moreover,  $\Omega(\Omega(A)) = \Omega(A)$  for each finite set  $A$  and  $\Omega(U) = U$  for each  $U \in \text{Fin}(X)_0$ .
- (2) The minimal iterator  $\mathbb{I}_0^{\text{Fin}}$  is a Peano iterator.
- (3) There exists a unique mapping  $[[\cdot]] : X \rightarrow \text{Fin}(X)_0$  with  $[[x_0]] = \emptyset$  such that  $[[f^{\text{Fin}}(x)]] = [[x]] \cup \{x\}$  for all  $x \in X$ . Moreover,  $x \notin [[x]]$  for each  $x \in X$ .
- (4)  $\Omega(A) = [[\omega(A)]]$  for each finite set  $A$ .
- (5) A set  $A$  is finite if and only if  $A \approx [[x]]$  for some  $x \in X$ .
- (6) If  $A$  is a finite set and  $a \notin A$  then  $\Omega(A \cup \{a\})$  is the disjoint union of  $\Omega(A)$  and  $\{\omega(A)\}$ .



- (7) If  $B$  is a proper subset of a finite set  $A$  then  $\Omega(B)$  is a proper subset of  $\Omega(A)$ .  
 (8) If  $U, V \in \text{Fin}(X)_0$  with  $U \neq V$  then either  $U$  is a proper subset of  $V$  or  $V$  is a proper subset of  $U$ .

*Proof* (1) Let  $A$  be a finite set and let  $\mathcal{S} = \{B \subset A : \Omega(B) \approx B\}$  and so  $\emptyset \in \mathcal{S}$ , since  $\Omega(\emptyset) = \emptyset$ . Thus let  $B \in \mathcal{S}^p$ ,  $a \in A \setminus B$  and put  $B' = B \cup \{a\}$ . Then  $\Omega(B') = f^{\text{Fin}}(\Omega(B)) = \{x_0\} \cup f(\Omega(B))$ . But  $f(\Omega(B)) \approx \Omega(B) \approx B$ , since  $f$  is injective and  $B \in \mathcal{S}$ . Also  $x_0 \notin f(\text{Fin}(X))$  and this implies that  $\Omega(B') \approx B'$ , i.e.,  $B' \in \mathcal{S}$ . Hence  $\mathcal{S}$  is an inductive  $A$ -system, and therefore  $\Omega(A) \approx A$ . It follows immediately that  $\Omega(\Omega(A)) = \Omega(A)$  for each finite set  $A$  and  $\Omega(U) = U$  for each  $U \in \text{Fin}(X)_0$ .

(2) It now follows from (1) that the iterator  $\mathbb{I}^{\text{Fin}}$  is proper and hence that the minimal iterator  $\mathbb{I}_0^{\text{Fin}}$  is a Peano iterator.

(3) By the recursion theorem (Theorem 3.4) for the iterator  $\mathbb{I}$  applied to the iterator  $\mathbb{I}^{\text{Fin}}$  there exists a unique mapping  $[[\cdot]] : X \rightarrow \text{Fin}(X)$  with  $[[x_0]] = \emptyset$  and such that  $[[f(x)]] = f^{\text{Fin}}([x]) = \{x_0\} \cup f([x])$  for all  $x \in X$  and note that by Lemma 3.7  $[X] \subset \text{Fin}(X)_0$ . Next let  $X_1 = \{x \in X : [f(x)] = [x] \cup \{x\}\}$ . Then  $[[f(x_0)]] = \{x_0\} \cup f([x_0]) = \{x_0\} \cup f(\emptyset) = \emptyset \cup \{x_0\} = [x_0] \cup \{x_0\}$  and so  $x_0 \in X_1$ . Let  $x \in X_1$ ; then

$$\begin{aligned} [[f(f(x))]] &= \{x_0\} \cup f([f(x)]) = \{x_0\} \cup f([x] \cup \{x\}) \\ &= \{x_0\} \cup f([x]) \cup \{f(x)\} = [f(x)] \cup \{f(x)\} \end{aligned}$$

and so  $f(x) \in X_1$ . Thus  $X_1$  is an  $f$ -invariant subclass of  $X$  containing  $x_0$  and hence  $X_1 = X$ , i.e.,  $[[f(x)]] = [x] \cup \{x\}$  for all  $x \in X$ . The uniqueness of the mapping  $[[\cdot]] : X \rightarrow \text{Fin}(X)_0$  (satisfying  $[[x_0]] = \emptyset$  and  $[[f(x)]] = [x] \cup \{x\}$  for all  $x \in X$ ) follows immediately from the fact that  $\mathbb{I}$  is minimal.

Now consider the subclass  $X_2 = \{x \in X : x \notin [x]\}$ ; in particular  $x_0 \in X_2$ , since  $x_0 \notin \emptyset = [x_0]$ . If  $x \in X_2$  then  $f(x) \notin f([x])$  (since  $f$  is injective) and  $f(x) \notin \{x_0\}$  (since  $x_0 \notin f(X)$ ) and so  $f(x) \notin \{x_0\} \cup f([x]) = [f(x)]$ . Hence  $f(x) \in X_2$ . Thus  $X_2$  is an  $f$ -invariant subclass of  $X$  containing  $x_0$  and therefore  $X_2 = X$ , i.e.,  $x \notin [x]$  for all  $x \in X$ .

(4) Let  $A$  be a finite set and put  $\mathcal{S} = \{B \in \mathcal{P}(A) : \Omega(B) = [[\omega(B)]]\}$ . Then  $\emptyset \in \mathcal{S}$ , since  $[[\omega(\emptyset)]] = [[x_0]] = \emptyset = \Omega(\emptyset)$ . Thus consider  $B \in \mathcal{S}^p$  and let  $a \in A \setminus B$ . Then

$$[[\omega(B \cup \{a\})]] = [[f(\omega(B))]] = f^{\text{Fin}}([[\omega(B)]]) = f(\Omega(B)) = \Omega(B \cup \{a\})$$

and hence  $B \cup \{a\} \in \mathcal{S}$ . This shows  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Thus  $\Omega(A) = [[\omega(A)]]$  for each finite set  $A$ .

(5) If  $A$  is a finite set then  $x = \omega(A) \in X$  and by (1) and (4) it follows that  $[[x]] = [[\omega(A)]] = \Omega(A) \approx A$ . On the other hand,  $[[x]]$  is a finite set for each  $x \in X$  and so if  $A \approx [[x]]$  then  $A$  is finite.

(6) If  $A$  is a finite set and  $a \notin A$  then by (4)

$$\Omega(A \cup \{a\}) = [[\omega(A \cup \{a\})]] = [[f(\omega(A))]] = [[\omega(A)]] \cup \{\omega(A)\} = \Omega(A) \cup \{\omega(A)\}$$

and by (1)  $\omega(A) \notin \Omega(A)$ , since  $A \not\approx A \cup \{a\}$ .

(7) Let  $B$  be a subset of a finite set  $A$ , put  $C = A \setminus B$  and let  $\mathcal{S}$  denote the set of subsets  $D$  of  $C$  for which  $\Omega(B) \subset \Omega(B \cup D)$ . Then it follows immediately from (6) that  $\mathcal{S}$  is an inductive  $C$ -system and thus  $C \in \mathcal{S}$ , i.e.,  $\Omega(B) \subset \Omega(A)$ . Moreover, if  $B$  is a proper subset of  $A$  then by (1)  $\Omega(B)$  is a proper subset of  $\Omega(A)$ .

(8) By Lemma 2.2 there exists either a proper subset  $A$  of  $U$  with  $A \approx V$  or a proper subset  $B$  of  $V$  with  $B \approx U$ . Assume the former holds. Then  $\Omega(A) = \Omega(V)$  and by (7)  $\Omega(A)$  is a proper subset of  $\Omega(U)$  and hence by (1)  $\Omega(V) = V$  is a proper subset of  $\Omega(U) = U$ . If the latter holds then the same argument shows that  $U$  is a proper subset of  $V$ .  $\square$

If Theorem 3.5 is applied to the Peano iterator  $(\mathbb{N}, s, 0)$  then it is easy to see that  $[[n]] = \{0, 1, \dots, n-1\}$  for each  $n \in \mathbb{N}$ .

Theorem 3.5 will now be employed to obtain a result which guarantees the existence of mappings 'defined by recursion'.

**Theorem 3.6** *Let  $\mathbb{I} = (X, f, x_0)$  be a Peano iterator, let  $Y$  and  $Z$  be classes and let  $\beta : Y \rightarrow Z$  and  $\alpha : X \times Y \times Z \rightarrow Z$  be mappings. Then there is a unique mapping  $\pi : X \times Y \rightarrow Z$  with  $\pi(x_0, y) = \beta(y)$  for all  $y \in Y$  such that  $\pi(f(x), y) = \alpha(x, y, \pi(x, y))$  for all  $x \in X, y \in Y$ .*

*Proof* The notation is as in Theorem 3.5. Let  $X_0$  be the subclass of  $X$  consisting of those  $x \in X$  for which there exist a unique mapping  $\pi_x : [[f(x)]] \times Y \rightarrow Z$  with  $\pi_x(x_0, y) = \beta(y)$  for all  $y \in Y$  and  $\pi_x(f(x'), y) = \alpha(x', y, \pi(x', y))$  for all  $x' \in [[f(x)]]$ ,  $y \in Y$ . Now  $[[f(x_0)]] = [[x_0]] \cup \{x_0\} = \emptyset \cup \{x_0\} = \{x_0\}$ . Thus  $\pi_{x_0} : \{x_0\} \times Y \rightarrow Z$  has to be defined by  $\pi_{x_0}(x_0, y) = \beta(y)$ , which shows that  $x_0 \in X_0$ . Thus let  $x \in X_0$  with unique mapping  $\pi_x : [[f(x)]] \times Y \rightarrow Z$ . Then  $[[f(f(x))]]$  is the disjoint union of  $[[f(x)]]$  and  $\{f(x)\}$  and so we have to define  $\pi_{f(x)} : [[f(f(x))]] \times Y \rightarrow Z$  by letting  $\pi_{f(x)}(x', y) = \pi_x(x', y)$  if  $x' \in [[f(x)]]$  and  $\pi_{f(x)}(f(x), y) = \alpha(x, y, \pi(x, y))$  for all  $y \in Y$ . It follows that  $f(x) \in X_0$  and hence  $X_0$  is an  $f$ -invariant subclass of  $X$  containing  $x_0$ . Hence  $X_0 = X$ . Now define

$\pi : X \times Y \rightarrow Z$  with by letting  $\pi(x, y) = \pi_x(x, y)$  for all  $x \in X$ ,  $y \in Y$ . Then  $\pi(x_0, y) = \pi_{x_0}(x_0, y) = \beta(y)$  for all  $y \in Y$  and if  $x \in X$  then

$$\begin{aligned}\pi(f(x), y) &= \pi_{f(x)}(f(x), y) = \alpha(x, y, \pi_{\pi(x)}(x, y)) \\ &= \alpha(x, y, \pi_x(x, y)) = \alpha(x, y, \pi(x, y))\end{aligned}$$

for all  $y \in Y$ . Finally, if  $\pi'$  is another mapping satisfying the conditions of the theorem then it easy to see that  $\{x \in X : \pi(x, y) = \pi'(x, y) \text{ for all } y \in Y\}$  is a  $f$ -invariant subclass of  $X$  containing  $x_0$  and hence  $X_0 = X$ , i.e.,  $\pi' = \pi$ . Therefore the mapping  $\pi$  is unique.  $\square$

The following is a special case of Theorem 3.6 in which the class  $Y$  consists of a single element (and so it can be omitted from the statement of the theorem): Let  $\mathbb{I} = (X, f, x_0)$  be a Peano iterator, let  $Z$  be a class,  $\beta_0 \in Z$  and  $\alpha : X \times Z \rightarrow Z$  be a mapping. Then there is a unique mapping  $\pi : X \rightarrow Z$  with  $\pi(x_0) = \beta_0$  such that  $\pi(f(x)) = \alpha(x, \pi(x))$  for all  $x \in X$ . This is a common way of defining a mapping by primitive recursion.

Before going any further we need to be more explicit about the structure preserving mappings between iterators. In the following let  $\mathbb{I} = (X, f, x_0)$  and  $\mathbb{J} = (Y, g, y_0)$  be iterators; a mapping  $\mu : X \rightarrow Y$  is called a *morphism* from  $\mathbb{I}$  to  $\mathbb{J}$  if  $\mu(x_0) = y_0$  and  $g \circ \mu = \mu \circ f$ . This will also be expressed by saying that  $\mu : \mathbb{I} \rightarrow \mathbb{J}$  is a morphism. The recursion theorem thus states that if  $\mathbb{I}$  is a Peano iterator then for each iterator  $\mathbb{J}$  there exists a unique morphism  $\mu : \mathbb{I} \rightarrow \mathbb{J}$ .

**Lemma 3.6** (1) *The identity mapping  $\text{id}_X$  is a morphism from  $\mathbb{I}$  to  $\mathbb{I}$ .*

(2) *Let  $\mathbb{K} = (Z, h, z_0)$  be a further iterator. If  $\mu : \mathbb{I} \rightarrow \mathbb{J}$  and  $\nu : \mathbb{J} \rightarrow \mathbb{K}$  are morphisms then  $\nu \circ \mu$  is a morphism from  $\mathbb{I}$  to  $\mathbb{K}$ .*

*Proof* (1) This is clear, since  $\text{id}_X(x_0) = x_0$  and  $f \circ \text{id}_X = f = \text{id}_X \circ f$ .

(2) This follows since  $(\nu \circ \mu)(x_0) = \nu(\mu(x_0)) = \nu(y_0) = z_0$  and

$$\beta \circ (\nu \circ \mu) = (\beta \circ \nu) \circ \mu = (\nu \circ g) \circ \mu = \nu \circ (g \circ \mu) = \nu \circ (\mu \circ f) = (\nu \circ \mu) \circ f. \quad \square$$

If  $\mu : \mathbb{I} \rightarrow \mathbb{J}$  is a morphism then clearly  $\mu \circ \text{id}_X = \mu = \text{id}_Y \circ \mu$ , and if  $\mu, \nu$  and  $\tau$  are morphisms for which the compositions are defined then  $(\tau \circ \nu) \circ \mu = \tau \circ (\nu \circ \mu)$ .

**Lemma 3.7** (1) *If  $\mathbb{I}$  is minimal then there is at most one morphism  $\mu : \mathbb{I} \rightarrow \mathbb{J}$ .*

(2) *If  $\mathbb{J}$  is minimal and  $\mu : \mathbb{I} \rightarrow \mathbb{J}$  is a morphism then  $\mu$  is surjective.*

(3) *If  $\mathbb{I}$  is minimal and  $\mu : \mathbb{I} \rightarrow \mathbb{J}$  is a morphism then  $\mu(X) = Y_0$ , where  $Y_0$  is the least  $g$ -invariant subclass of  $Y$  containing  $y_0$ .*

*Proof* (1) Let  $\mu, \mu' : \mathbb{I} \rightarrow \mathbb{J}$  be morphisms and let  $X_0 = \{x \in X : \mu(x) = \mu'(x)\}$ . Then  $x_0 \in X_0$ , since  $\mu(x_0) = \nu'(x_0) = y_0$  and if  $x \in X_0$  then

$$\mu(f(x)) = g(\mu(x)) = g(\mu'(x)) = \mu'(f(x))$$

and therefore  $f(x) \in X_0$ . Hence  $X_0$  is an  $f$ -invariant subclass of  $X$  containing  $x_0$  and so  $X_0 = X$ , i.e.,  $\mu = \mu'$ .

(2) Let  $Y_0 = \{\mu(x) : x \in X\}$ . Then  $y_0 = \mu(x_0) \in Y_0$  and if  $h = \mu(x) \in Y_0$  then  $g(h) = g(\mu(x)) = \mu(g(x)) \in Y_0$ . Thus  $Y_0$  is a  $g$ -invariant subclass of  $Y$  containing  $y_0$  and hence  $Y_0 = Y$ . This shows  $\mu$  is surjective.

(3) Let  $X_0 = \{x \in X : \mu(x) \in Y_0\}$ . Then  $x_0 \in X_0$ , since  $\mu(x_0) = y_0 \in Y_0$ , and if  $x \in X_0$  then  $\mu(f(x)) = g(\mu(x)) \in Y_0$ , since  $Y_0$  is  $g$ -invariant. Therefore  $X_0$  is a  $f$ -invariant subclass of  $X$  containing  $x_0$  and hence  $X_0 = X$ , i.e.,  $\mu(X) \subset Y_0$ . Now since  $\mu(X) \subset Y_0$  we can consider  $\mu$  as a morphism  $\mu_0$  from  $\mathbb{I}$  to  $\mathbb{J}_0$ , where  $\mathbb{J}_0$  is the corresponding minimal iterator, and by (2)  $\mu_0$  is surjective. But this implies that  $\mu(X) = Y_0$ .  $\square$

The iterators  $\mathbb{I}$  and  $\mathbb{J}$  are said to be *isomorphic* if there exists a morphism  $\mu : \mathbb{I} \rightarrow \mathbb{J}$  and a morphism  $\nu : \mathbb{J} \rightarrow \mathbb{I}$  such that  $\nu \circ \mu = \text{id}_X$  and  $\mu \circ \nu = \text{id}_Y$ . In particular, the mappings  $\mu$  and  $\nu$  are then both bijections.

**Lemma 3.8** *If  $\mu : \mathbb{I} \rightarrow \mathbb{J}$  is a morphism and the mapping  $\mu : X \rightarrow Y$  is a bijection then the inverse mapping  $\mu^{-1} : Y \rightarrow X$  is a morphism from  $\mathbb{J}$  to  $\mathbb{I}$  and so  $\mathbb{I}$  and  $\mathbb{J}$  are isomorphic.*

*Proof* We have  $g = g \circ \mu \circ \mu^{-1} = \mu \circ f \circ \mu^{-1}$  and so  $\mu^{-1} \circ g = \mu^{-1} \circ \mu \circ f \circ \mu^{-1} = f \circ \mu^{-1}$ . Thus  $\mu^{-1}$  is a morphism from  $\mathbb{J}$  to  $\mathbb{I}$ , since also  $\mu^{-1}(y_0) = x_0$ .  $\square$

The iterator  $\mathbb{I}$  is said to be *initial* if for each iterator  $\mathbb{J}$  there is a unique morphism from  $\mathbb{I}$  to  $\mathbb{J}$ . Theorem 3.3 (the recursion theorem) thus states that a Peano iterator is initial.

**Lemma 3.9** *(Let  $\mathbb{I}$  be initial and  $\pi : \mathbb{I} \rightarrow \mathbb{J}$  be the unique morphism.*

*(1) If  $\mathbb{J}$  is initial then  $\pi$  is an isomorphism and so  $\mathbb{I}$  and  $\mathbb{J}$  are isomorphic.*

*(2) If  $\pi$  is an isomorphism then  $\mathbb{J}$  is initial.*

*Proof* (1) There exists a unique morphism  $\tau : \mathbb{J} \rightarrow \mathbb{I}$  (since  $\mathbb{J}$  is initial). Thus  $\tau \circ \pi : \mathbb{I} \rightarrow \mathbb{I}$  is a morphism. But  $\text{id}_X : \mathbb{I} \rightarrow \mathbb{I}$  is also a morphism and there is a

unique morphism from  $\mathbb{I}$  to  $\mathbb{I}$  (since  $\mathbb{I}$  is initial) and hence  $\tau \circ \pi = \text{id}_X$ . In the same way  $\pi \circ \tau = \text{id}_Y$ . Therefore  $\pi$  is an isomorphism and so  $\mathbb{I}$  and  $\mathbb{J}$  are isomorphic.

(2) Let  $\mathbb{K}$  be an iterator and  $\mu : \mathbb{I} \rightarrow \mathbb{K}$  be the unique morphism. Then  $\mu \circ \pi^{-1}$  is a morphism from  $\mathbb{J}$  to  $\mathbb{K}$ . If  $\nu : \mathbb{J} \rightarrow \mathbb{K}$  is any morphism then  $\nu \circ \pi$  is a morphism from  $\mathbb{I}$  to  $\mathbb{K}$  and thus  $\nu \circ \pi = \mu$ . Hence  $\nu = \mu \circ \pi^{-1}$  and so there is a unique morphism from  $\mathbb{J}$  to  $\mathbb{K}$ , which shows that  $\mathbb{J}$  is initial.  $\square$

Lemma 3.9 shows that, up to isomorphism, there is a unique initial iterator. Of course, this is only true if a initial iterator exists, and  $(\mathbb{N}, \mathbf{s}, 0)$  is an initial iterator. In fact, in Section 4 we will exhibit another initial iterator  $\mathbb{O}_0 = (O, \sigma_0, \emptyset)$  which is defined only in terms of finite sets and makes no use of the infinite set  $\mathbb{N}$ . The elements of  $O$  are the finite ordinals.

The following result of Lawvere [5] shows that the converse of the recursion theorem holds.

**Theorem 3.7** *An initial iterator  $\mathbb{I} = (X, f, x_0)$  is a Peano iterator.*

*Proof* We first show that  $\mathbb{I}$  is minimal, and then that it is  $\mathbb{N}$ -like.

**Lemma 3.10** *The initial iterator  $\mathbb{I}$  is minimal.*

*Proof* Let  $X_0 = \{x \in X : x = \omega(A) \text{ for some finite set } A\}$  of  $X$ , let  $f_0 : X_0 \rightarrow X_0$  be the restriction of  $f$  to  $X_0$ . Then the iterator  $\mathbb{I}_0 = (X_0, f_0, x_0)$  is minimal and the inclusion mapping  $\text{inc} : X_0 \rightarrow X$  defines a morphism from  $\mathbb{I}_0$  to  $\mathbb{I}$ . Let  $\mu : \mathbb{I} \rightarrow \mathbb{I}_0$  be the unique morphism; then  $\text{inc} \circ \mu = \text{id}_X$ , since by Lemma 3.6  $\text{inc} \circ \mu$  and  $\text{id}_X$  are both morphisms from  $\mathbb{I}$  to  $\mathbb{I}$  (and there is only one such morphism, since  $\mathbb{I}$  is initial). In particular,  $\text{inc}$  is surjective, which implies that  $X_0 = X$ , i.e.,  $\mathbb{I}$  is minimal.  $\square$

**Lemma 3.11** *The initial iterator  $\mathbb{I}$  is  $\mathbb{N}$ -like.*

*Proof* Let  $\diamond$  be an element not contained in  $X$ , put  $X_\diamond = X \cup \{\diamond\}$  and define a mapping  $f_\diamond : X_\diamond \rightarrow X_\diamond$  by putting  $f_\diamond(x) = f(x)$  for  $x \in X$  and  $f_\diamond(\diamond) = x_0$ ; thus  $\mathbb{I}_\diamond = (X_\diamond, f_\diamond, \diamond)$  is an iterator. Since  $\mathbb{I}$  is initial there exists a unique morphism  $\mu : \mathbb{I} \rightarrow \mathbb{I}_\diamond$ . Consider the subclass  $X' = \{x \in X : f_\diamond(\mu(x)) = x\}$ ; then  $x_0 \in X'$ , since  $f_\diamond(\mu(x_0)) = f_\diamond(\diamond) = x_0$  and if  $x \in X'$  then  $f_\diamond(\mu(x)) = x$  and so

$$f_\diamond(\mu(f(x))) = f_\diamond(f_\diamond(\mu(x))) = f_\diamond(x) = f(x) ,$$

i.e.,  $f(x) \in X'$ . Thus  $X'$  is a  $f$ -invariant subclass of  $X$  containing  $x_0$  and hence  $X' = X$ , since by Lemma 3.10  $\mathbb{I}$  is minimal. Thus  $\mu(f(x)) = f_\diamond(\mu(x)) = x$  for all  $x \in X$ , which implies that  $f$  is injective. Moreover,  $x_0 \notin f(X)$ , since

$$\nu(f(x)) = f_\diamond(\mu(x)) \neq \diamond = \mu(x_0)$$

for all  $x \in X$ . Hence  $\mathbb{I}$  is  $\mathbb{N}$ -like.  $\square$

This completes the proof of Theorem 3.7.  $\square$

We end the section with a couple of remarks about the case when  $\mathbb{I} = (X, f, x_0)$  is a Peano iterator with  $X$  an infinite set. A set  $E$  is defined to be *Dedekind-infinite* if there exists an injective mapping  $h : E \rightarrow E$  which is not surjective, and so by Theorem 2.1 a Dedekind-infinite set is infinite, i.e., it is not finite. The converse also holds (i.e., every infinite set is Dedekind-infinite) provided a suitable form of the axiom of choice is assumed. In models without the axiom of choice there can exist infinite sets which are Dedekind-finite. If  $(X, f, x_0)$  is a Peano iterator then the set  $X$  is Dedekind-infinite. Conversely, if  $E$  is a Dedekind-infinite set and  $h : E \rightarrow E$  is injective but not surjective and  $e_0 \in E \setminus h(E)$  then  $(E_0, y_0, e_0)$  is a Peano iterator, where  $E_0$  is the least  $h$ -invariant subset of  $E$  containing  $e_0$  and  $y_0 : E_0 \rightarrow E_0$  is the restriction of  $h$  to  $E_0$ . Thus a Peano iterator whose first component is a set exists if and only if there exists a Dedekind-infinite set.

The following somewhat strange result can be found in [11] and in [9].

**Proposition 3.5** *Suppose that there exists a Dedekind-infinite set (for example, the set of natural numbers  $\mathbb{N}$  is a such a set). Then a set  $A$  is finite if and only if  $\mathcal{P}(\mathcal{P}(A))$  is Dedekind-finite.*

*Proof* Let  $\mathbb{I} = (X, f, x_0)$  be a Peano iterator (which exists since a Dedekind-infinite set exists) and let  $\omega$  be the assignment of finite sets in  $\mathbb{I}$ . Now let  $E$  be an infinite set and consider the mapping  $h : \mathcal{P}(E) \rightarrow X$  given by  $h(A) = \omega(A)$  if  $A$  is finite, and  $h(Y) = x_0$  if  $Y$  is infinite. If  $A$  is finite then by Proposition 2.12 there exists a finite subset  $C$  of  $E$  with  $C \approx A$  and thus by Lemma 3.5  $h$  is surjective. Hence there exists an injective mapping  $g : \mathcal{P}(X) \rightarrow \mathcal{P}(\mathcal{P}(E))$ . (If  $f : Y \rightarrow Z$  is any surjective mapping then the mapping  $g : \mathcal{P}(Z) \rightarrow \mathcal{P}(Y)$  given by  $g(E) = f^{-1}(E)$  for each  $E \in \mathcal{P}(Z)$  is injective.) There is then an injective mapping  $\alpha : X \rightarrow \mathcal{P}(\mathcal{P}(E))$  given by  $\alpha(x) = g(\{x\})$  for all  $x \in X$ . Therefore by Lemma 3.12 below  $\mathcal{P}(\mathcal{P}(E))$  is Dedekind-infinite. On the other hand, if  $E$  is finite then by Proposition 2.3  $\mathcal{P}(E)$  and hence  $\mathcal{P}(\mathcal{P}(E))$  is finite, and so by Theorem 2.1  $\mathcal{P}(\mathcal{P}(E))$  is Dedekind-finite. Thus if there exists a Dedekind-infinite set then a set  $A$  is finite if and only if  $\mathcal{P}(\mathcal{P}(A))$  is Dedekind-finite.  $\square$

Note that the assumption about the Dedekind-infinite set is only needed to show that if  $Y$  is infinite then  $\mathcal{P}(\mathcal{P}(Y))$  is Dedekind-infinite. But if  $Y$  is infinite then in fact it can be shown that the class  $O$  of finite ordinals is actually a set (and thus a Dedekind-infinite set) and so the hypothesis is not actually needed.

**Lemma 3.12** *A set containing a Dedekind-infinite set is itself Dedekind-infinite.*

*Proof* Let  $E$  contain a Dedekind-infinite set  $F$  and so there exists an injective mapping  $h : F \rightarrow F$  which is not surjective. Define  $g : E \rightarrow E$  by letting  $g(x) = h(x)$  if  $x \in F$  and  $g(x) = x$  if  $x \in E \setminus F$ . Then  $g$  is injective but not surjective and hence  $E$  is Dedekind-infinite.  $\square$

## 4 Finite ordinals

In this section we study finite ordinals using the standard approach introduced by von Neumann [10]. As in Section 3 we denote the class of all finite sets by  $\mathbf{Fin}$ . Let  $\sigma : \mathbf{Fin} \rightarrow \mathbf{Fin}$  be the mapping given by  $\sigma(A) = A \cup \{A\}$  for each finite set  $A$ . If we iterate the operation  $\sigma$  starting with the empty set and label the resulting sets using the natural numbers then we obtain the following:

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= \sigma(0) = 0 \cup \{0\} = \emptyset \cup \{0\} = \{0\}, \\ 2 &= \sigma(1) = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\}, \\ 3 &= \sigma(2) = 2 \cup \{2\} = \{0, 1\} \cup \{2\} = \{0, 1, 2\}, \\ 4 &= \sigma(3) = 3 \cup \{3\} = \{0, 1, 2\} \cup \{3\} = \{0, 1, 2, 3\}, \\ 5 &= \sigma(4) = 4 \cup \{4\} = \{0, 1, 2, 3\} \cup \{4\} = \{0, 1, 2, 3, 4\}, \\ n+1 &= \sigma(n) = n \cup \{n\} = \{0, 1, 2, \dots, n-1\} \cup \{n\} = \{0, 1, 2, \dots, n\}. \end{aligned}$$

Denote by  $\mathbb{O}$  the iterator  $(\mathbf{Fin}, \sigma, \emptyset)$ . Then by Theorem 3.1 there exists a unique assignment  $\varrho$  of finite sets in  $\mathbb{O}$ . Thus  $\varrho : \mathbf{Fin} \rightarrow \mathbf{Fin}$  is the unique mapping with  $\varrho(\emptyset) = \emptyset$  and such that

$$\varrho(A \cup \{a\}) = \sigma(\varrho(A))$$

for each finite set  $A$  and each element  $a \notin A$ . Moreover, if  $A$  and  $B$  are finite sets with  $A \approx B$  then  $\varrho(A) = \varrho(B)$ .

**Theorem 4.1** *For each finite set  $A$  we have  $\varrho(A) \approx A$ , and thus  $\varrho(A) = \varrho(B)$  if and only if  $A \approx B$ .*

*Proof* Let  $A$  be a finite set and let  $\mathcal{S} = \{B \subset A : \varrho(B) \approx B\}$ . In particular  $\emptyset \in \mathcal{S}$ , since  $\varrho(\emptyset) = \emptyset$ . Thus let  $B \in \mathcal{S}$ ,  $a \in A \setminus B$  and put  $B' = B \cup \{a\}$ . Then  $\varrho(B) \approx B$  and hence  $\varrho(B') = \sigma(\varrho(B)) \approx B'$ . Thus  $B' \in \mathcal{S}$  and so  $\mathcal{S}$  is an inductive  $A$ -system. Therefore  $A \in \mathcal{S}$ , i.e.,  $\varrho(A) \approx A$ .  $\square$

It follows from Theorem 4.1 that  $\varrho(\varrho(A)) = \varrho(A)$  for each finite set  $A$ .

Let  $O = \{B \in \mathbf{Fin} : B = \varrho(A) \text{ for some finite set } A\}$ . The elements of  $O$  will be called *finite ordinals*. Thus for each finite set  $A$  there exists a unique  $o \in O$  with  $o = \varrho(A)$ . By Lemma 3.4  $O$  is the least  $\sigma$ -invariant subclass of  $\mathbf{Fin}$  containing  $\emptyset$ . Let  $\sigma_0 : O \rightarrow O$  be the restriction of  $\sigma$  to  $O$ . Then  $\mathbb{O}_0 = (O, \sigma_0, \emptyset)$  is a minimal iterator.

We do not assume that  $O$  is a set but if it were then it would not be finite. (If  $O$  were finite then  $o' = \sigma(\varrho(O))$  would be a finite ordinal, but  $o' \not\approx o$  for each  $o \in O$ .)



**Theorem 4.2**  $\mathbb{O}_0$  is a Peano iterator. Therefore by Theorem 3.3 (the recursion theorem) it follows that for each iterator  $\mathbb{J} = (H, \delta, h_0)$  there exists a unique mapping  $\pi : O \rightarrow H$  with  $\pi(\emptyset) = h_0$  such that  $\pi \circ \sigma = \delta \circ \pi$ .

*Proof* By Theorem 4.1  $\mathbb{O}$  is proper and hence also  $\mathbb{O}_0$  is proper. Therefore by Theorem 3.2  $\mathbb{O}_0$  is a Peano iterator.  $\square$

The iterator  $\mathbb{O}_0$  is obtained without making use of the natural numbers or any other infinite set. There are versions of set theory in which every set is finite. In such a set theory there is no set of natural numbers and so the iterator  $(\mathbb{N}, \mathbf{s}, 0)$  does not exist. However, the Peano iterator  $\mathbb{O}_0$  does exist and in this case  $O$  is not a set. If  $(\mathbb{N}, \mathbf{s}, 0)$  does exist then by Lemma 3.9 it is isomorphic to  $\mathbb{O}_0$ . Thus  $\mathbb{O}_0$  can be considered as a particular version of  $(\mathbb{N}, \mathbf{s}, 0)$  and we can use the usual notation for the elements of  $\mathbb{N}$  to denote the elements of  $O$ . In Section 6 we show how the arithmetic operations of addition, multiplication and exponentiation can be introduced for any minimal iterator. In particular, this can be applied to the iterator  $\mathbb{O}_0$ .

**Proposition 4.1** For each finite set  $A$

$$\varrho(A) = \{o \in O : o = \varrho(A') \text{ for some proper subset } A' \text{ of } A\}.$$

*Proof* Let  $A$  be a finite set and denote by  $\mathcal{S}$  the set of subsets  $B$  of  $A$  for which

$$\varrho(B) = \{o \in O : o = \varrho(B') \text{ for some proper subset } B' \text{ of } B\}.$$

In particular  $\emptyset \in \mathcal{S}$ . Thus consider  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ . Then

$$\varrho(B \cup \{a\}) = \sigma(\varrho(B)) = \varrho(B) \cup \{\varrho(B)\} = \{o \in O : o = \varrho(B') \text{ for some } B' \subset B\}.$$

But if  $C$  is a proper subset of  $B \cup \{a\}$  then  $C \approx C'$  for some  $C' \subset C$  and then by Theorem 3.1  $\varrho(C) = \varrho(C')$ . It follows that

$$\varrho(B \cup \{a\}) = \{o \in O : o = \varrho(B') \text{ for some proper subset } B' \text{ of } B \cup \{a\}\}$$

and thus  $B \cup \{a\} \in \mathcal{S}$ . Hence  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Therefore

$$\varrho(A) = \{o \in O : o = \varrho(A') \text{ for some proper subset } A' \text{ of } A\}. \quad \square$$

**Proposition 4.2** Let  $A$  be a finite set and  $B \subset A$ ; then  $\varrho(B) \subset \varrho(A)$ .

*Proof* This follows immediately from Proposition 4.1.  $\square$

Let  $o \in O$  and let  $a$  be an element not in  $o$ . Then  $\varrho(o \cup \{a\}) = \sigma(o)$ .

**Proposition 4.3** *For each  $o \in O$  we have*

$$o = \{o' \in O : o' \text{ is a proper subset of } o\},$$

$$\sigma(o) = \{o' \in O : o' \text{ is a subset of } o\}.$$

*Proof* This follows from Proposition 4.1.  $\square$

A set  $E$  is said to be *totally ordered with respect to set membership* if, whenever  $e_1$  and  $e_2$  are distinct elements of  $E$  then exactly one of  $e_1 \in e_2$  and  $e_2 \in e_1$  holds. By Proposition 4.3 each finite ordinal  $\alpha$  is totally ordered with respect to set inclusion and, moreover, each element of  $\alpha$  is a subset of  $\alpha$ .

In the general (non-finite case) the usual definition of an ordinal is as a set having these two properties [10].

**Proposition 4.4** *Let  $o, o' \in O$  with  $o \neq o'$ . Then either  $o$  is a proper subset of  $o'$  or  $o'$  is a proper subset of  $o$ .*

*Proof* By Theorem 4.1  $o \not\approx o'$  and so by Lemma 2.2 there either exists a proper subset  $B$  of  $o$  with  $B \approx o'$  or there exists a proper subset  $B'$  of  $o'$  with  $B' \approx o$ . Suppose the former holds. Then  $\varrho(B) = o'$  and therefore by Proposition 4.1

$$o' = \varrho(B) = \{b : b = \varrho(B') \text{ for some proper subset } B' \text{ of } B\},$$

$$o = \varrho(o) = \{b : b = \varrho(B') \text{ for some proper subset } B' \text{ of } o\}.$$

It follows that  $o'$  is a proper subset of  $o$ . If the latter holds then, in the same way,  $o$  is a proper subset of  $o'$ .  $\square$

If  $o, o' \in O$  then we write  $o' \leq o$  if  $o' \subset o$ . By Proposition 4.4  $\leq$  defines a total order on  $O$ . As usual we write  $o' < o$  to denote that  $o' \neq o$  and  $o' \leq o$ .

**Lemma 4.1** *Each non-empty subclass  $O'$  of  $O$  contains a minimum element, i.e., an element  $o$  with  $o \leq o'$  for all  $o' \in O'$ .*

*Proof* Let  $p \in O'$ ; then  $\{o' \in O' : o' \leq p\}$  is a non-empty finite totally ordered set and hence by Proposition 2.14 it contains a minimal element  $o$ . Thus  $o \leq o'$  for all  $o' \in O'$ .  $\square$

The following induction principle for finite ordinals corresponds to Theorem 2.3.

**Proposition 4.5** *Let  $P$  be a statement about finite ordinals. Suppose  $P(0)$  holds and that  $P(\sigma(o))$  holds whenever  $P(o)$  holds for  $o \in O$ . Then  $P$  is a property of finite ordinals, i.e.,  $P(o)$  holds for every  $o \in O$ .*

*Proof* Let  $A$  be a finite set and put  $\mathcal{S} = \{B \in \mathcal{P}(A) : P(\varrho(B)) \text{ holds}\}$ . Then  $P(\emptyset) = P(0)$  holds, so let  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ . Then  $\varrho(b \cup \{a\}) = \sigma(\varrho(B))$  and thus  $B \cup \{a\} \in \mathcal{S}$ . Therefore  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Let  $o \in O$ ; then  $o$  is a finite set and hence applying the above with  $A = o$  shows that  $P(o)$  holds.  $\square$

We end this section by looking at a two further Peano iterators  $\mathbb{U}_0$  and  $\mathbb{V}_0$ . They have nothing to do with ordinals, except that the iterator  $\mathbb{O}_0$  is involved in their definition. What they have in common with the iterator  $\mathbb{O}_0$  is that they are defined 'absolutely' and are in fact constructed solely from operations performed on the empty set  $\emptyset$ . First consider the iterator  $\mathbb{V} = (\mathbf{Fin}, \beta, \emptyset)$ , where the mapping  $\beta : \mathbf{Fin} \rightarrow \mathbf{Fin}$  is given by  $\beta(A) = \{A\}$  for each finite set  $A$ . The recursion theorem for the Peano iterator  $\mathbb{O}_0$  applied to the iterator  $\mathbb{V}$  implies there exists a unique morphism  $\eta : \mathbb{O}_0 \rightarrow \mathbb{V}$ . Thus  $\eta : O \rightarrow \mathbf{Fin}$  is the unique mapping with  $\eta(0) = \emptyset$  such that  $\eta(\sigma(o)) = \beta(\eta(o))$  for each  $o \in O$ . If we iterate the operation  $\beta$  starting with the empty set then we obtain the following:

$$\begin{aligned} \eta(0) &= \emptyset, \\ \eta(1) &= \beta(\eta(0)) = \{\emptyset\}, \\ \eta(2) &= \beta(\eta(1)) = \{\{\emptyset\}\}, \\ \eta(3) &= \beta(\eta(2)) = \{\{\{\emptyset\}\}\}, \\ \eta(4) &= \beta(\eta(3)) = \{\{\{\{\emptyset\}\}\}\}, \\ &\vdots \eta(n) = \{\{\{\{\{\dots\{\emptyset\}\dots\}\}\}\}\}. \end{aligned}$$

Let  $\varphi$  be the assignment of finite sets in  $\mathbb{V}$ , thus  $\varphi : \mathbf{Fin} \rightarrow \mathbf{Fin}$  is the unique mapping with  $\varphi(\emptyset) = \emptyset$  such that  $\varphi(A \cup \{a\}) = \beta(\varphi(A))$  for each finite set  $A$  and each  $a \notin A$ , and by Theorem 3.1  $\varphi(A) = \varphi(B)$  whenever  $A \approx B$ . Let  $V = \{v \in \mathbf{Fin} : v = \varphi(A) \text{ for some finite set } A\}$ , so by Lemma 3.4  $V$  is the least  $\beta$ -invariant subclass of  $\mathbf{Fin}$  containing  $\emptyset$ ; let  $\beta_0 : V \rightarrow V$  be the restriction of  $\beta$  to  $V$ . Thus  $\mathbb{V}_0 = (V, \beta_0, \emptyset)$  is a minimal iterator.

**Lemma 4.2** *For each finite set  $A$  we have  $\varphi(A) = \eta(\varrho(A))$ .*

*Proof* Let  $A$  be a finite set and let  $\mathcal{S} = \{B \in \mathcal{P}(A) : \varphi(B) = \eta(\varrho(B))\}$  and in particular  $\emptyset \in \mathcal{S}$ , since  $\varphi(\emptyset) = \eta(\varrho(\emptyset)) = \emptyset$ . Thus let  $B \in \mathcal{S}$  and  $b \in A \setminus B$ . Then  $\varphi(B \cup \{b\}) = \beta(\varphi(B)) = \beta(\eta(\varrho(B))) = \eta(\sigma(\varrho(B))) = \eta(\varrho(B \cup \{b\}))$  and hence  $B \cup \{b\} \in \mathcal{S}$ . This shows that  $\mathcal{S}$  is an inductive  $A$ -system and therefore  $A \in \mathcal{S}$ , i.e.,  $\varphi(A) = \eta(\varrho(A))$ .  $\square$

**Lemma 4.3** *The mapping  $\eta : O \rightarrow \text{Fin}$  maps  $O$  bijectively onto  $V$ .*

*Proof* By Lemma 3.7  $\eta(O) = V$  and so it remains to show that  $\eta$  is injective. Suppose this is not the case and let

$$O_0 = \{o \in O : \text{there exists } o' \in O \text{ with } o < o' \text{ and } \eta(o) = \eta(o')\}.$$

Thus  $O_0$  is non-empty and so by Lemma 4.1 it contains a minimum element  $o_0$  with  $o_0 < o$  for all  $o \in O_0$ , and since  $o_0 \in O_0$  there exists  $o_1 \in O_0$  with  $o_0 < o_1$  and  $\eta(o_0) = \eta(o_1)$ . Now if  $p \in O \setminus \{0\}$  then by Proposition 3.3 there exists a unique  $q \in O$  with  $p = \sigma(q)$  and then  $\eta(p) = \eta(\sigma(q)) = \beta(\eta(q)) = \{\eta(q)\}$ . Thus if  $p \in O \setminus \{0\}$  then there exists a unique  $q \in O$  with  $\eta(p) = \{\eta(q)\}$ . In particular,  $\eta(p) \neq \emptyset$  if  $p \neq 0$  and so  $o_1 \neq 0$ , since  $\eta(0) = \emptyset$  and  $o_2 \neq 0$ . There thus exist unique  $q_1, q_2 \in O$  with  $\eta(o_1) = \{\eta(q_1)\}$  and  $\eta(o_2) = \{\eta(q_2)\}$ . Then  $\{\eta(q_1)\} = \{\eta(q_2)\}$  and so  $\eta(q_1) = \eta(q_2)$ . But  $q_1 < o_1$  and  $q_1 < q_2$ , which contradicts the minimality of  $o_1$ . Therefore  $\eta$  is injective.  $\square$

**Proposition 4.6** *The iterator  $\mathbb{V}_0$  is a Peano iterator.*

*Proof* By Lemma 4.3 and Lemma 3.6 (3) the morphism  $\eta$  is an isomorphism and by the recursion theorem  $\mathbb{O}_0$  is initial. Thus by Lemma 3.9 (2)  $\mathbb{V}_0$  is initial and therefore by Theorem 3.7  $\mathbb{V}_0$  is a Peano iterator.  $\square$

Except for being a Peano iterator the iterator  $\mathbb{V}_0$  has none of the properties enjoyed by  $\mathbb{O}_0$ . It corresponds to the perhaps most primitive method of counting by representing the number  $n$  with something like  $n$  marks, in this case the empty set enclosed in  $n$  braces.

We can improve the situation somewhat by applying Theorem 3.5 to the iterator  $\mathbb{V}_0 = (V, \beta_0, \emptyset)$ . This results in the iterator  $\mathbb{U} = (\text{Fin}, \alpha, \emptyset)$ , where the mapping  $\alpha : \text{Fin} \rightarrow \text{Fin}$  is given by  $\alpha(A) = \{\emptyset\} \cup \beta(A)$  for each finite set  $A$ , and where to simplify things we use  $\text{Fin}$  instead of  $\text{Fin}(V)$  as the first component of  $\mathbb{U}$ .

Let  $\psi : \mathbf{Fin} \rightarrow \mathbf{Fin}$  be the assignment of finite sets in  $\mathbb{U}$  and  $\mathbb{U}_0 = (U, \alpha_0, \emptyset)$  be the corresponding minimal iterator, so  $U$  is the least  $\alpha$ -invariant subclass of  $\mathbf{Fin}$  containing  $\emptyset$  and  $\alpha_0 : U \rightarrow U$  is the restriction of  $\alpha$  to  $U$ . Therefore by Lemma 3.4

$$U = \{u \in \mathbf{Fin} : u = \psi(A) \text{ for some finite set } A\}.$$

Theorem 3.5 now states that

- (1)  $\psi(A) \approx A$  for each finite set  $A$ . Thus if  $A$  and  $B$  are finite sets then  $A \approx B$  if and only if  $\psi(A) = \psi(B)$ . Moreover,  $\psi(\psi(A)) = \psi(A)$  for each finite set  $A$  and  $\psi(u) = u$  for each  $u \in U$ .
- (2) The minimal iterator  $\mathbb{U}_0$  is a Peano iterator.
- (3) There exists a unique mapping  $[[\cdot]] : V \rightarrow U$  with  $[[\emptyset]] = \emptyset$  and such that  $[[\beta(v)]] = [[v]] \cup \{v\}$  for all  $v \in V$ . Moreover,  $v \notin [[v]]$  for each  $v \in V$ .
- (4)  $\psi(A) = [[\varphi(A)]]$  for each finite set  $A$ .
- (5) A set  $A$  is finite if and only if  $A \approx [[v]]$  for some  $v \in V$ .
- (6) If  $A$  is a finite set and  $a \notin A$  then  $\psi(A \cup \{a\})$  is the disjoint union of  $\psi(A)$  and  $\{\varphi(A)\}$ .
- (7) If  $B$  is a proper subset of a finite set  $A$  then  $\psi(B)$  is a proper subset of  $\psi(A)$ .
- (8) If  $u, v \in U$  with  $u \neq v$  then either  $u$  is a proper subset of  $v$  or  $v$  is a proper subset of  $u$ .

Note that the following holds for the mapping  $[[\cdot]]$ :

$$\begin{aligned} [[\emptyset]] &= \emptyset, \\ [[\{\emptyset\}]] &= \{\emptyset\}, \\ [[\{\{\emptyset\}\}]] &= \{\emptyset, \{\emptyset\}\}, \\ [[\{\{\{\emptyset\}\}\}]] &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\ [[\{\{\{\{\emptyset\}\}\}\}]] &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}\}, \\ &\vdots \\ [[\eta(n+1)]] &= \{\eta(0), \eta(1), \dots, \eta(n)\}. \end{aligned}$$

## 5 Finite minimal iterators

Theorems 3.2 and 3.4 imply that for a minimal iterator  $\mathbb{I} = (X, f, x_0)$  there are two mutually exclusive possibilities: Either  $\mathbb{I}$  is a Peano iterator or  $X$  is a finite set. In this section we deal with case in which  $X$  is a finite set.

Thus in what follows let  $\mathbb{I} = (X, f, x_0)$  be a minimal iterator with  $X$  a finite set. For each  $x \in X$  let  $X_x$  be the least  $f$ -invariant subset of  $X$  containing  $x$  and let  $f_x : X_x \rightarrow X_x$  be the restriction of  $f$  to  $X_x$ . Thus  $\mathbb{I}_x = (X_x, f_x, x)$  is a minimal iterator. Also let  $\omega_x$  be the unique assignment of finite sets in  $\mathbb{I}_x$ ; put  $\omega = \omega_{x_0}$ .

The iterator  $\mathbb{I}$  can be considered as a finite dynamical system with the dynamics given by the mapping  $f : X \rightarrow X$  and with initial state  $x_0$ . Now it is an elementary fact that the mapping  $f$  is then eventually periodic. What this means is usually defined in terms of the iterates of  $x$  under  $f$  and these are most easily defined in terms of the natural numbers: Let  $x \in X$ ; then applying the recursion theorem for the iterator  $(\mathbb{N}, s, 0)$  to the iterator  $\mathbb{I}_x$  there exists a unique mapping  $\pi_x : \mathbb{N} \rightarrow X_x$  with  $\pi_x(0) = x$  such that  $\pi_x(n+1) = f_x(\pi_x(n))$  for all  $n \in \mathbb{N}$ . We use the standard notation and write  $f^n(x)$  instead of  $\pi_x(n)$ . The element  $f^n(x)$  is the  $n$ -th iterate of  $f$  when the initial state is  $x$ .

An element  $x \in X$  is periodic if  $x = f^n(x)$  for some  $n \geq 1$  and if  $n$  is the least such index then the set  $\{f^k(x) : 0 \leq k < n\}$  is the corresponding periodic cycle. The elementary fact about  $\mathbb{I}$  states that there is a unique periodic cycle and that there exists  $m \geq 0$  such that  $f^n(x_0)$  is periodic for all  $n \geq m$ . Note that the uniqueness of the periodic cycle only holds because  $\mathbb{I}$  is minimal.

We will obtain this result without making use of the natural numbers or of any other Peano iterator and so we need a new definition of being periodic which does not involve the iterates of  $f$ . It is straightforward to check that the following is equivalent to the above definition: An element  $x \in X$  is said to be *periodic* if  $x \in f_x(X_x)$  and so by Proposition 3.3 and Theorem 2.1  $x$  is periodic if and only if  $f_x$  is a bijection.

**Theorem 5.1** (1) *Let  $X_P = \{x \in X : x \text{ is periodic}\}$ . Then  $X_P$  is non-empty and  $X_x = X_y$  for all  $x, y \in X_P$ . Thus  $f$  maps  $X_P$  bijectively onto itself.*

(2) *Let  $X_N = \{x \in X : x \text{ is not periodic}\}$  and suppose  $X_N \neq \emptyset$ . Then  $f$  is injective on  $X_N$  and there exists a unique element  $u \in X_N$  such that  $f(u)$  is periodic. Moreover, there exists a unique element  $v \in X_P$  such that  $f(v) = f(u)$  and  $u$  and  $v$  are the unique elements of  $X$  with  $u \neq v$  such that  $f(u) = f(v)$ .*

For the proof of (1) we need the following:

**Lemma 5.1** *Let  $x \in X$ . Then:*

- (1)  $X_{f(x)} \subset X_x \subset \{x\} \cup X_{f(x)}$  and  $X_{f(x)} \subset f(X_x)$ .
- (2) If  $x$  is periodic then so is  $f(x)$ .
- (3)  $x$  is periodic if and only if  $X_{f(x)} = X_x$ . Moreover, if  $x$  is not periodic then  $X_{f(x)} = X_x \setminus \{x\}$  and so  $X_{f(x)}$  is a proper subset of  $X_x$ .
- (4) If  $x$  is periodic then  $y$  is periodic and  $X_y = X_x$  for all  $y \in X_x$ .
- (5) If  $A$  is a finite set,  $B \subset A$  and  $x = \omega(B)$  then  $\omega(A) \in X_x$ .
- (6) For each  $x \in X$  we have  $X_y \subset X_x$  for all  $y \in X_x$ .

*Proof* (1)  $X_x$  is  $f$ -invariant and contains  $f(x)$  and hence  $X_{f(x)} \subset X_x$ . Also  $\{x\} \cup X_{f(x)}$  is  $f$ -invariant and contains  $x$  and therefore  $X_x \subset \{x\} \cup X_{f(x)}$ . Moreover,  $f(X_x)$  is  $f$ -invariant and contains  $f(x)$  and so  $X_{f(x)} \subset f(X_x)$ .

(2) Let  $x$  be periodic; then  $f_x : X_x \rightarrow X_x$  is a bijection. Thus  $f_{f(x)} : X_{f((x))} \rightarrow X_{f(x)}$  is injective, since by (1)  $X_{f(x)} \subset X_x$ . Hence  $f_{f(x)} : X_{f((x))} \rightarrow X_{f(x)}$  is a bijection and so  $f(x)$  is periodic.

(3) Suppose  $X_{f(x)} = X_x$ ; then  $x \in X_x = X_{f(x)}$  and so by (1)  $x \in f(X_x)$ . Thus by Proposition 3.3  $x$  is periodic. Suppose conversely that  $x$  is periodic. Then, as in (2)  $f_{f(x)} : X_{f((x))} \rightarrow X_{f(x)}$  and  $f_x : X_x \rightarrow X_x$  are both bijections. But by (1)  $X_x$  is either  $X_{f(x)}$  or  $X_{f(x)} \cup \{x\}$  and it follows that  $X_{f(x)} = X_x$ , since if  $x \notin X_{f(x)}$  then  $f(x) = x$  which would imply that  $X_{f(x)} = X_x$ . If  $x$  is not periodic then  $X_{f(x)} \neq X_x$  and so by (1)  $X_{f(x)} = X_x \setminus \{x\}$ .

(4) Let  $\mathcal{S} = \{y \in X : y \text{ is periodic and } X_y = X_x\}$  then  $x \in \mathcal{S}$  and if  $y \in \mathcal{S}$  then by (2)  $f(y)$  is periodic and by (3)  $X_{f(y)} = X_y = X_x$ . Thus  $f(y) \in \mathcal{S}$  and so  $\mathcal{S}$  is  $f$ -invariant. Hence  $\mathcal{S} \subset X_x$ , i.e.,  $y$  is periodic and  $X_y = X_x$  for all  $y \in X_x$ .

(5) Let  $C = A \setminus B$  and put  $\mathcal{S} = \{D \subset C : \omega(D) \in X_x\}$ . Then  $\emptyset \in \mathcal{S}$  and if  $D \in \mathcal{S}^p$  and  $d \in C \setminus D$  then  $\omega(D \cup \{d\}) = f(\omega(D)) \in \mathcal{S}$ , since  $\omega(D) \in \mathcal{S}$  and  $X_x$  is  $f$ -invariant. Therefore  $\mathcal{S}$  is an inductive  $C$  system and thus  $C \in \mathcal{S}$ , i.e.,  $A = B \cup C \in X_x$ .

(6) Let  $X' = \{y \in X : X_y \subset X_x\}$ . Then  $x \in X'$  and if  $y \in X'$  (and so  $X_y \subset X_x$ ) then  $X_{f(y)} \subset X_y \subset X_x$  and hence  $f(y) \in X'$ . Therefore  $X_x \subset X'$ , i.e.,  $X_y \subset X_x$  for all  $y \in X_x$ .  $\square$

*Proof of Theorem 5.1 (1)* Let  $\mathcal{S} = \{X_y : y \in X\}$ . Then by Proposition 1.2 there exists  $x \in X$  such that  $X_x$  is a minimal element of  $\mathcal{S}$ . But by Lemma 5.1 (1)  $X_{f(x)} \subset X_x$  and so  $X_{f(x)} = X_x$ . Hence by Lemma 5.1 (2)  $x$  is periodic, i.e.,  $x \in X_P$ . Thus by Lemma 5.1 (4)  $y$  is periodic and  $X_y = X_x$  for all  $y \in X_x$ .

Let  $x, y \in X_P$ ; by Lemma 3.5 there exist finite sets  $A$  and  $B$  with  $x = \omega(A)$  and  $y = \omega(B)$  and by Lemma 2.3 and Theorem 3.1 we can assume that  $B \subset A$  or  $A \subset B$  and without loss of generality assume that  $B \subset A$ . Put  $C = A \setminus B$  and let  $\mathcal{S} = \{D \subset C : \omega(B \cup D) \text{ is periodic and } X_{\omega(B \cup D)} = X_y\}$  and so  $\emptyset \in \mathcal{S}$ . Thus let  $D \in \mathcal{S}$  and  $d \in C \setminus D$ . Then  $\omega(B \cup D \cup \{d\}) = f(z)$  where  $z = \omega(B \cup D)$  and  $z$  is periodic and  $X_z = X_y$ , since  $D \in \mathcal{S}$ . Hence by Lemma 5.1 (2) and (3)  $f(z)$  is periodic and  $X_{f(z)} = X_z = X_y$ , which shows that  $D \cup \{d\} \in \mathcal{S}$ . Therefore  $\mathcal{S}$  is an inductive  $C$ -system and so  $C \in \mathcal{S}$ , i.e.,  $X_x = X_y$ .  $\square$

For the proof of (2) we need the following:

**Lemma 5.2** *Let  $s, t \in X$  with  $s \neq t$  and  $f(s) = f(t)$ . Then  $u = f(s) = f(t)$  is periodic. Moreover, one of  $s$  and  $t$  is periodic.*

*Proof* By Lemma 3.5 there exist finite sets  $B$  and  $C$  such that  $s = \omega(B)$  and  $t = \omega(C)$  and by Theorem 3.1 and Lemma 2.3 we can assume without loss of generality that  $B \subset C$ , and so  $B$  is a proper subset of  $C$ . Let  $d \notin C$  and put  $B' = B \cup \{d\}$ ,  $C' = C \cup \{d\}$ . Then  $\omega(B') = \omega(C') = u$ . Now let  $a \in C \setminus B$  and put  $C'' = C' \setminus \{a\}$ . But  $u = \omega(B')$  and  $B' \subset C''$  and so Lemma 5.1 (5) implies that  $\omega(C'') \in X_u$  and then  $u = f(\omega(C'')) \in f(X_u)$ . This shows that  $u$  is periodic. Now  $B$  is a proper subset of  $C$  and so there exists  $D \supset B'$  with  $D \approx C$ . Thus  $u = \omega(B')$  and  $t = \omega(C) = \omega(D)$  and  $B' \subset D$  and so by Lemma 5.1 (5)  $t \in X_u$ . Therefore by Lemma 5.1 (4)  $t$  is periodic.  $\square$

*Proof of Theorem 5.1 (2)* Lemma 5.2 implies that  $f$  is injective on  $X_N$ . By Proposition 1.2 there exists  $u \in X_N$  such that  $X_u$  is a minimal element of the set  $\mathcal{S}_N = \{X_y : y \in X_N\}$ . Then  $u$  is not periodic and so by Lemma 5.1 (1) and (3)  $X_{f(u)}$  is a proper subset of  $X_u$  and hence  $X_{f(u)} \notin \mathcal{S}_N$ , i.e.,  $f(u)$  is periodic. Suppose there exist  $u_1, u_2 \in X_N$  with  $u_1 \neq u_2$  and such that  $f(u_1)$  and  $f(u_2)$  are both periodic. Then there exist finite sets  $A_1, A_2$  with  $u_i = \omega(A_i)$  for  $i = 1, 2$  and as usual we can assume that  $A_1$  is a proper subset of  $A_2$ . Let  $a \in A_2 \setminus A_1$  and so  $A'_1 = A_1 \cup \{a\} \subset A_2$ . Then  $\omega(A'_1) = f(\omega(A_1)) = f(u_1)$  is periodic and by Lemma 5.1 (5)  $u_2 \in X_{f(u_1)}$ . Hence by Lemma 5.1 (3)  $u_2$  would be periodic. This contradiction shows that there is a unique  $u \in X_N$  such that  $z = f(u)$  is periodic. Thus  $z \in f_z(X_z)$  and so there exists a finite set  $A'$  such that  $z = f(\omega_{v'}(A'))$ . Let  $a$  be an element not in  $A'$  and put  $A = A' \cup \{a\}$ . Then  $A$  is a non-empty finite set and  $z = \omega_z(A)$ . Let  $v = \omega_v(A')$ . Then  $v \in X_v$  and so  $v$  is periodic and  $f(v) = z = f(u)$ . Moreover,  $v$  is the unique element of  $X_P$  with  $f(v) = f(u)$ , since  $f$  maps  $X_P$  bijectively onto itself. Finally, let  $u', v' \in X$  with  $u' \neq v'$  and  $f(u') = f(v')$ . Since  $f$  is injective on  $X_N$  and on  $X_P$  one of these elements is in  $X_N$  and the other in  $X_P$ . Label them so  $u' \in X_N$  and  $v' \in X_P$ . Then by



*Lemma 5.2*  $f(u') \in X_P$  and by the uniqueness of  $u$  it follows that  $u' = u$  and by the uniqueness of  $v$  it follows that  $v' = v$ .  $\square$

We next consider the special case in which  $X$  contains a fixed-point, i.e., an element  $z$  with  $f(z) = z$ . If  $z = x_0$  then  $X = \{x_0\}$  and we assume that this is not the case, and thus  $z \neq x_0$ . Since  $z$  is periodic it follows from Theorem 5.1 (1) that  $X_P = X_z = \{z\}$ . Hence  $X_N = X \setminus \{z\}$  and by Theorem 5.1 (2)  $f$  is injective on  $X_N$  and there exists a unique  $w \in X_N$  such that  $f(w) = z$ .

Let  $A$  be a finite set with  $z = \omega(A)$  and by Proposition 1.2 we can assume that  $z \neq \omega(B)$  for each proper subset  $B$  of  $A$ .

**Lemma 5.3** For each  $x \in X$  there exists  $B \subset A$  with  $x = \omega(B)$ .

*Proof* Let  $X_0 = \{x \in X : x = \omega(B) \text{ for some } B \subset A\}$ , and thus  $x_0 \in X_0$ , since  $x_0 = \omega(\emptyset)$ . Let  $x \in X_0$  with  $x = \omega(B)$ . If  $B$  is a proper subset of  $A$  and  $a \in A \setminus B$  then  $B \cup \{a\} \subset A$  and  $f(x) = \omega(B \cup \{a\})$  and so  $f(x) \in X_0$ . But if  $B = A$  then  $x = z$  and so  $f(z) = z \in X_0$ . Thus  $X_0$  is  $f$ -invariant and contains  $x_0$  and hence  $X_0 = X$ .  $\square$

**Lemma 5.4** If  $B, B' \in \mathcal{P}(A)$  with  $\omega(B) = \omega(B')$  then  $B \approx B'$ .

*Proof* Suppose there exist  $B, B' \in \mathcal{P}(A)$  with  $\omega(B) = \omega(B')$  and  $B \not\approx B'$ . Then by Lemma 2.2 (and if necessary exchanging the rôles of  $B$  and  $B'$ ) there exist such  $B, B'$  with  $B' \subset B$ , i.e., with  $B'$  a proper subset of  $B$ . Let  $\mathcal{S}$  be the subset of  $\mathcal{P}(A)$  consisting of those subsets  $B$  which contain a proper subset  $B'$  with  $\omega(B) = \omega(B')$ . Thus  $\mathcal{S}$  is non-empty and hence by Proposition 1.3  $\mathcal{S}$  contains a maximal element  $C$ ; let  $C'$  be a proper subset of  $C$  with  $\omega(C) = \omega(C')$ . Now  $C \neq A$ , since otherwise  $C'$  would be a proper subset of  $A$  with  $\omega(C') = z$ . Choose  $a \in A \setminus C$ ; then  $C' \cup \{a\}$  is a proper subset of  $C \cup \{a\}$ . But

$$\omega(C' \cup \{a\}) = f(\omega(C')) = f(\omega(C)) = \omega(C \cup \{a\}),$$

which contradicts the maximality of  $C$ . Therefore  $B \approx B'$  whenever  $B, B'$  are subsets of  $A$  with  $\omega(B) = \omega(B')$ .  $\square$

**Lemma 5.5** If  $x_1, x_2 \in X$  with  $X_{x_1} = X_{x_2}$  then  $x_1 = x_2$ .

*Proof* Let  $x_1, x_2 \in X$  with  $x_1 \neq x_2$ . Then by Lemma 5.3 and Lemma 2.2 (and if necessary exchanging the rôles of  $x_1$  and  $x_2$ ) there exist  $B_1, B_2 \in \mathcal{P}(A)$  with  $x_k = \omega(B_k)$  for  $k = 1, 2$  and such that  $B_1$  is a proper subset of  $B_2$ . Let  $b \in B_2 \setminus B_1$ ; then  $B'_1 = B_1 \cup \{a\} \subset B_2$  and  $\omega(B'_1) = f(x_1)$ . Thus by Lemma 5.1 (5)  $x_2 \in X_{f(x_1)}$  and so  $X_{x_2} \subset X_{f(x_1)}$ . If  $x_1 = z$  then  $X_{f(x_1)} = \{z\}$  and so  $X_{x_2} = \{z\}$ . But  $x_2 \neq z$ , and Lemma 5.1 (3) implies that  $X_{f(x_2)}$  would be a proper subset of  $X_{x_2} = \{z\}$ . Therefore this case cannot occur. Hence  $x_1 \neq z$  and by Lemma 5.1 (3)  $X_{f(x_1)}$  is a proper subset of  $X_{x_1}$ . Therefore  $X_{x_1} \neq X_{x_2}$ .  $\square$

Define a relation  $\leq$  on  $X$  by stipulating that  $x_1 \leq x_2$  if  $X_{x_2} \subset X_{x_1}$ . As usual,  $x_1 < x_2$  means that  $x_1 \leq x_2$  but  $x_1 \neq x_2$ .

**Proposition 5.1** (1) The relation  $\leq$  is a total order on  $X$  with  $x_0 \leq x \leq z$  for all  $x \in X$ . Moreover, if  $x \in X \setminus \{z\}$  then  $x < f(x)$ .

(2) If  $x \leq y \leq f(x)$  then  $y = x$  or  $y = f(x)$ .

(3) Let  $x, y \in X$ . if  $x \leq y$  if then  $f(x) \leq f(y)$ . Moreover, if  $x, y \in X \setminus \{z\}$  and  $f(x) \leq f(y)$  then  $x \leq y$ .

(4) Let  $x, y \in X \setminus \{z\}$ . Then  $x < y$  if and only if  $f(x) < f(y)$ .

*Proof* (1) It is clear that  $\leq$  is transitive and if both  $x_1 \leq x_2$  and  $x_2 \leq x_1$  hold then by Lemma 5.5  $x_1 = x_2$ . Let  $x_1, x_2 \in X$  and as usual we can assume that there exist  $B_1, B_2 \in \mathcal{P}(A)$  with  $x_k = \omega(B_k)$  for  $k = 1, 2$  and either  $B_2 \subset B_1$  or  $B_1 \subset B_2$ . If  $B_2 \subset B_1$  then by Lemma 5.1 (5)  $x_1 \in X_{x_2}$  and hence  $X_{x_1} \subset X_{x_2}$ . If  $B_1 \subset B_2$  then in the same way  $X_{x_2} \subset X_{x_1}$ . Therefore either  $x_1 \leq x_2$  or  $x_2 \leq x_1$  and this shows that  $\leq$  is a total order. It is clear that  $x_0 \leq x \leq z$  for all  $x \in X$ . If  $x \in X \setminus \{z\}$  then by Lemma 5.1 (3)  $X_{f(x)}$  is a proper subset of  $X_x$  and so  $x < f(x)$ .

(2) If  $x \leq y \leq f(x)$  then  $X_y \subset X_x$  and  $X_{f(x)} \subset X_y$  and hence

$$\{x\} \cup X_y \subset \{x\} \cup X_x = X_x = \{x\} \cup X_{f(x)} \subset \{x\} \cup X_y.$$

It follows that  $\{x\} \cup X_y = X_x = \{x\} \cup X_{f(x)} = \{x\} \cup X_y$  and in particular  $\{x\} \cup X_y = X_x$ . Thus either  $x \in X_y$ , in which case  $X_x = X_y$ , or  $x \notin X_y$ , in which case  $X_y = X_x \setminus \{x\}$  and then by Lemma 5.1 (3)  $X_y = X_{f(x)}$ . Therefore by Lemma 5.5 either  $y = x$  or  $y = f(x)$ .

(3) Note that for all  $x \in X$  both  $x \leq z$  and  $f(x) \leq f(z)$  hold trivially. Thus we can assume that  $y \in X \setminus \{z\}$ . Suppose  $x \leq y$  and so also  $x \in X \setminus \{z\}$ . Then  $X_y \subset X_x$  and  $X_{f(y)} = X_y \setminus \{y\}$ ,  $X_{f(x)} = X_x \setminus \{x\}$  and thus  $X_{f(y)} \subset X_{f(x)}$  provided  $x \notin X_y$ . But if  $x \in X_y$  then  $X_x \subset X_y$  and so  $X_x = X_y$ . Hence either  $f(x) \leq f(y)$  or  $x = y$  and in both cases  $f(x) \leq f(y)$ . Next let  $x, y \in X \setminus \{z\}$  with  $f(x) \leq f(y)$ .

Then either  $x \leq y$  or  $y \leq x$  and if  $y \leq x$  then by the above  $f(y) \leq f(x)$  and so  $f(x) = f(y)$ . But if  $f(x) = f(y)$  then  $x = y$ , since  $f$  is injective on  $X \setminus \{z\}$ , and again  $x \leq y$ .

(4) If  $x < y$  then by (3)  $f(x) \leq f(y)$  and  $f(x) \neq f(y)$  since  $x \neq y$  and  $f$  is injective on  $X \setminus \{z\}$ . Hence  $f(x) < f(y)$ . In the same way, if  $f(x) < f(y)$  then by (3)  $x \leq y$ . But  $x \neq y$  since  $f(x) \neq f(y)$  and thus  $x < y$ .  $\square$

The construction given above can be reversed. Let  $(X, \leq)$  be a finite totally ordered set with least element  $x_0$  and greatest element  $z$ . Define a mapping  $f : X \rightarrow X$  by letting  $f(x)$  be the least element in  $\{y \in X : y > x\}$  if  $x \neq z$  and putting  $f(z) = z$ . Then  $\mathbb{I} = (X, f, x_0)$  is a finite iterator with fixed point  $z$ .

**Proposition 5.2** *The iterator  $\mathbb{I}$  is minimal.*

*Proof* Let  $X_0$  be an  $f$ -invariant subset of  $X$  containing  $x_0$  and suppose  $X_0 \neq X$ . Let  $u$  be the least element in  $X \setminus X_0$  and so  $u \neq x_0$ . Let  $v$  be the greatest element in  $\{y \in X : y < u\}$ . Then  $v < u$ , hence  $v \in X_0$  and so  $f(v) \in X_0$ . But this is not possible, since  $f(v) = u$ . Therefore  $X_0 = X$  which shows that  $\mathbb{I}$  is minimal.  $\square$

Let  $G$  be a finite group, with the product of  $a$  and  $b$  in  $G$  denoted just by  $ab$ , with identity element  $1$  and with  $a^{-1}$  the inverse of  $a$ . For each  $a \in G$  let  $n_a : G \rightarrow G$  be given by  $n_a(b) = ab$  for each  $b \in G$ . Then there is the iterator  $\mathbb{I}_a = (G, n_a, 1)$ . Also let  $G_a$  be the least  $n_a$ -invariant subset of  $G$  containing  $1$  and  $m_a : G_a \rightarrow G_a$  be the restriction of  $n_a$  to  $G_a$ , so  $(G_a, m_a, 1)$  is a minimal iterator.

**Proposition 5.3** *For each  $a \in G$  the mapping  $m_a$  is a bijection and so  $a$  is periodic.*

*Proof* Suppose  $a$  is not periodic. Then by Theorem 5.1 (2) there exists a unique non-periodic element  $u \in G_a$  such that  $m_a(u)$  is periodic and a unique periodic element  $v \in G_a$  with  $m_a(v) = m_a(u)$ , i.e.,  $av = au$ . But then  $v = u$ , which contradicts the fact that  $u$  is not periodic and  $v$  is periodic. Hence  $a$  is periodic and  $m_a$  is a bijection.  $\square$

**Proposition 5.4** *Let  $H$  be a subset of  $G$  containing  $1$  and such that  $ab \in H$  for all  $a, b \in H$ . Then  $H$  is a subgroup of  $G$ .*

*Proof* Let  $a \in H$ ; then  $H$  is an  $m_a$ -invariant subset of  $G_a$  containing  $1$  and so  $G_a \subset H$ . Now by Proposition 5.3  $m_a : G_a \rightarrow G_a$  is a bijection and  $1 \in G_a$  and so there exists  $b \in G_a$  with  $m_a(b) = ab = 1$ . Thus  $b \in H$  and  $b = a^{-1}$ . This shows that  $H$  is a subgroup of  $G$ .  $\square$

## 6 Addition and multiplication

*In this section we show how an addition and a multiplication can be defined for any minimal iterator. These operations are associative and commutative and can be specified by the rules (a0), (a1), (m0) and (m1) below, which are usually employed when defining the operations on  $\mathbb{N}$  via the Peano axioms.*

*Note that, even if we do not assume the existence of an infinite set, we can apply the results of this section to the Peano iterator  $\mathbb{O}_0$ .*

*In the following let  $\mathbb{I} = (X, f, x_0)$  be a minimal iterator with  $\omega$  the assignment of finite sets in  $\mathbb{I}$ . We first state the main results (Theorems 6.1 and 6.2) and then develop the machinery required to prove them. In the following section we give alternative proofs for these theorems.*

**Theorem 6.1** *There exists a unique binary operation  $\oplus$  on  $X$  such that*

$$\omega(A) \oplus \omega(B) = \omega(A \cup B)$$

*whenever  $A$  and  $B$  are disjoint finite sets. This operation  $\oplus$  is both associative and commutative,  $x \oplus x_0 = x$  for all  $x \in X$  and for all  $x_1, x_2 \in X$  there is an  $x \in X$  such that either  $x_1 = x_2 \oplus x$  or  $x_2 = x_1 \oplus x$ . Moreover,  $\oplus$  is the unique binary operation  $\oplus$  on  $X$  such that*

$$(a0) \quad x \oplus x_0 = x \text{ for all } x \in X.$$

$$(a1) \quad x \oplus f(x') = f(x \oplus x') \text{ for all } x, x' \in X.$$

**Theorem 6.2** *There exists a unique binary operation  $\otimes$  on  $X$  such that*

$$\omega(A) \otimes \omega(B) = \omega(A \times B)$$

*for all finite sets  $A$  and  $B$ . This operation  $\otimes$  is both associative and commutative,  $x \otimes x_0 = x_0$  for all  $x \in X$  and  $x \otimes f(x_0) = x$  for all  $x \in X$  with  $x \neq x_0$  (and so  $f(x_0)$  is a multiplicative identity element) and the distributive law holds for  $\oplus$  and  $\otimes$ :*

$$x \otimes (x_1 \oplus x_2) = (x \otimes x_1) \oplus (x \otimes x_2)$$

*for all  $x, x_1, x_2 \in X$ . Moreover,  $\otimes$  is the unique binary operation on  $X$  such that*

$$(m0) \quad x \otimes x_0 = x_0 \text{ for all } x \in X.$$

$$(m1) \quad x \otimes f(x') = x \oplus (x \otimes x') \text{ for all } x, x' \in X.$$

We now prepare for the proofs of Theorems 6.1 and Theorem 6.2 and first look at what is common to these two theorems. Let  $D$  be a subset of  $\mathbf{Fin} \times \mathbf{Fin}$  and let  $\gamma : D \rightarrow \mathbf{Fin}$  be a mapping. In Theorem 6.1 we will have

$$D = \{(A, B) \in \mathbf{Fin} \times \mathbf{Fin} : A \text{ and } B \text{ are disjoint}\}$$

and  $\gamma(A, B) = A \cup B$  and in Theorem 6.2  $D = \mathbf{Fin} \times \mathbf{Fin}$  and  $\gamma(A, B) = A \times B$ . We assume that the mapping  $\omega' : D \rightarrow X \times X$  with  $\omega'(A, B) = (\omega(A), \omega(B))$  is surjective. By Lemma 3.5 this is clearly the case for Theorem 6.2 and for Theorem 6.1 it follows from the next result.

**Lemma 6.1** *For all  $(x, x') \in X \times X$  there exist disjoint finite sets  $A$  and  $B$  with  $(x, x') = (\omega(A), \omega(B))$ .*

*Proof* By Lemma 3.5 there exists  $(C, D) \in \mathbf{Fin} \times \mathbf{Fin}$  with  $(\omega(C), \omega(D)) = (x, x')$  and by Lemma 2.4 there exist disjoint finite sets  $A$  and  $B$  with  $A \approx C$  and  $B \approx D$ . Hence by Theorem 3.1 (2)  $A$  and  $B$  are disjoint with  $(x, x') = (\omega(A), \omega(B))$ .  $\square$

Theorem 6.1 and Theorem 6.2 state that for the appropriate mapping  $\gamma : D \rightarrow \mathbf{Fin}$  there exists a binary operation  $\odot$  on  $X$  such that  $\omega(A) \odot \omega(B) = \omega(\gamma(A, B))$  for all  $(A, B) \in D$ .

**Proposition 6.1** *Let  $\gamma : D \rightarrow \mathbf{Fin}$  be an arbitrary mapping for which the mapping  $\omega' : D \rightarrow X \times X$  is surjective and let  $\lambda : D \rightarrow X$  be the mapping defined by  $\lambda(A, B) = \omega(\gamma(A, B))$  for all  $(A, B) \in D$ . Then there exists a binary operation  $\odot$  on  $X$  such that  $\omega(A) \odot \omega(B) = \omega(\gamma(A, B))$  for all  $(A, B) \in D$  if and only if*

$$(\heartsuit) \quad \lambda(A, B) = \lambda(A', B') \text{ whenever } (A, B) \text{ and } (A', B') \text{ are elements of } D \text{ with } \omega'(A, B) = \omega'(A', B').$$

Moreover, if  $(\heartsuit)$  holds then  $\odot$  is the unique binary operation  $\odot'$  on  $X$  such that  $\omega(A) \odot' \omega(B) = \omega(\gamma(A, B))$  for all  $(A, B) \in D$ .

*Proof* This is a special case of Proposition 3.1. If  $\odot$  is written as a prefix operation then the requirement on  $\odot$  is that  $\odot(\omega'(A, B)) = \lambda(A, B)$  for all  $(A, B) \in D$  which in turn is the requirement that  $\odot \circ \omega' = \lambda$ .  $\square$

Let  $D_{\oplus} = \{(A, B) \in \mathbf{Fin} \times \mathbf{Fin} : A \text{ and } B \text{ are disjoint}\}$  and let  $\gamma_{\oplus} : D_{\oplus} \rightarrow \mathbf{Fin}$  be given by  $\gamma_{\oplus}(A, B) = A \cup B$  for all  $(A, B) \in D_{\oplus}$ . Also let  $D_{\otimes} = \mathbf{Fin} \times \mathbf{Fin}$  and let  $\gamma_{\otimes} : D_{\otimes} \rightarrow \mathbf{Fin}$  be given by  $\gamma_{\otimes}(A, B) = A \times B$  for all  $(A, B) \in D_{\otimes}$ . We will establish the existence of the operations  $\oplus$  and  $\otimes$  in Theorems 6.1 and 6.2 by showing that the mappings  $\gamma_{\oplus}$  and  $\gamma_{\otimes}$  satisfy condition  $(\heartsuit)$  and then applying Proposition 6.1.

For a Peano iterator  $\mathbb{I}$  this is not a problem. Consider  $\gamma_{\oplus}$ : If  $(A, B), (A', B') \in D_{\oplus}$  with  $\omega'(A, B) = \omega'(A', B')$  then by Theorem 3.2  $A \approx A'$  and  $B \approx B'$ , from which it easily follows that  $A \cup B \approx A' \cup B'$  and therefore by Theorem 3.1 (2) we have  $\omega(A \cup B) = \omega(A' \cup B')$ , i.e.,  $\omega(\gamma_{\oplus}(A, B)) = \omega(\gamma_{\oplus}(A', B'))$  and so  $\gamma_{\oplus}$  satisfies condition  $(\heartsuit)$ . Essentially the same proof also shows that  $\gamma_{\otimes}$  satisfies  $(\heartsuit)$ .

Once it is known that the operation  $\oplus$  exists then the remaining properties of  $\oplus$  listed in Theorem 6.1 follow from the corresponding properties of the union operation  $\cup$  (for example, that it is associative and commutative).

The following shows that  $\gamma_{\oplus}$  satisfies condition  $(\heartsuit)$ .

**Lemma 6.2** *If  $(A, B)$  and  $(A', B')$  are elements of  $D_{\oplus}$  with  $\omega'(A, B) = \omega'(A', B')$  then  $\omega(A \cup B) = \omega(A' \cup B')$ .*

*Proof* Consider finite sets  $A$  and  $A'$  with  $\omega(A) = \omega(A')$  and a finite set  $B$  disjoint from  $A$  and  $A'$ . Let  $\mathcal{S} = \{C \in \mathcal{P}(B) : \omega(A \cup C) = \omega(A' \cup C)\}$ . Then  $\emptyset \in \mathcal{S}$ , since  $\omega(A \cup \emptyset) = \omega(A) = \omega(A') = \omega(A' \cup \emptyset)$ . Let  $C \in \mathcal{S}$  and  $b \in B \setminus C$ . Then

$$\begin{aligned} \omega(A \cup (C \cup \{b\})) &= \omega((A \cup C) \cup \{b\}) = f(\omega(A \cup C)) \\ &= f(\omega(A' \cup C)) = \omega((A' \cup C) \cup \{b\}) = \omega(A' \cup (C \cup \{b\})) \end{aligned}$$

and hence  $C \cup \{b\} \in \mathcal{S}$ . Thus  $\mathcal{S}$  is an inductive  $B$ -system and so  $B \in \mathcal{S}$ . Therefore  $\omega(A \cup B) = \omega(A' \cup B)$ .

For any set  $C$  and any element  $d$  put  $C_d = C \times \{d\}$  (and so  $C_d \approx C$ ). Now let  $(A, B), (A', B') \in \gamma_{\oplus}$  with  $\omega'(A, B) = \omega'(A', B')$ , and choose distinct elements  $\triangleleft$  and  $\triangleright$ ; then  $\omega(A \cup B) = \omega(A_{\triangleleft} \cup B_{\triangleleft})$  (since  $A \cup B \approx A_{\triangleleft} \cup B_{\triangleleft}$ ),  $\omega(A'_{\triangleright} \cup B'_{\triangleright}) = \omega(A' \cup B')$  (since  $A'_{\triangleright} \cup B'_{\triangleright} \approx A' \cup B'$ ),  $\omega(A_{\triangleleft}) = \omega(A'_{\triangleright})$  (since  $A_{\triangleleft} \approx A$  and  $A' \approx A'_{\triangleright}$ ) and  $\omega(B_{\triangleleft}) = \omega(B'_{\triangleright})$  (since  $B_{\triangleleft} \approx B$  and  $B' \approx B'_{\triangleright}$ ), which gives us the following data:

- $\omega(A \cup B) = \omega(A_{\triangleleft} \cup B_{\triangleleft})$ ,
- $\omega(A_{\triangleleft}) = \omega(A'_{\triangleright})$  and  $B_{\triangleleft}$  is disjoint from both  $A_{\triangleleft}$  and  $A'_{\triangleright}$ ,
- $\omega(B_{\triangleleft}) = \omega(B'_{\triangleright})$  and  $A'_{\triangleright}$  is disjoint from both  $B_{\triangleleft}$  and  $B'_{\triangleright}$ ,
- $\omega(A'_{\triangleright} \cup B'_{\triangleright}) = \omega(A' \cup B')$ .

Thus by two applications of the first part of the proof

$$\begin{aligned} \omega(A \cup B) &= \omega(A_{\triangleleft} \cup B_{\triangleleft}) = \omega(A'_{\triangleright} \cup B_{\triangleleft}) \\ &= \omega(B_{\triangleleft} \cup A'_{\triangleright}) = \omega(B'_{\triangleright} \cup A'_{\triangleright}) = \omega(A'_{\triangleright} \cup B'_{\triangleright}) = \omega(A' \cup B') . \quad \square \end{aligned}$$

*Proof of Theorem 6.1:* By Lemma 6.2 condition  $(\heartsuit)$  holds for the mapping  $\gamma_\oplus$  and thus by Proposition 6.1 there exists a binary operation  $\oplus$  on  $X$  such that  $\omega(A) \oplus \omega(B) = \omega(A \cup B)$  whenever  $A$  and  $B$  are disjoint finite sets. Moreover,  $\oplus$  is the unique operation with this property. We show that  $\oplus$  is associative and commutative: Let  $x_1, x_2, x_3 \in X$ ; then by Lemma 3.5 there exist finite sets  $A_1, A_2$  and  $A_3$  with  $x_1 = \omega(A_1)$ ,  $x_2 = \omega(A_2)$  and  $x_3 = \omega(A_3)$  and by Proposition 2.9 and Theorem 3.1 (2) we can assume that these sets are disjoint. Clearly  $(A_1 \cup A_2) \cup A_3 \approx A_1 \cup (A_2 \cup A_3)$  and therefore

$$\begin{aligned} (x_1 \oplus x_2) \oplus x_3 &= (\omega(A_1) \oplus \omega(A_2)) \oplus \omega(A_3) = \omega(A_1 \cup A_2) \oplus \omega(A_3) \\ &= \omega((A_1 \cup A_2) \cup A_3) = \omega(A_1 \cup (A_2 \cup A_3)) \\ &= \omega(A_1) \oplus \omega(A_2 \cup A_3) = \omega(A_1) \oplus (\omega(A_2) \oplus \omega(A_3)) \\ &= x_1 \oplus (x_2 \oplus x_3) . \end{aligned}$$

In the same way  $\oplus$  is commutative. Let  $x_1, x_2 \in X$ ; then, as above there exist disjoint finite sets  $A_1$  and  $A_2$  with  $x_1 = \omega(A_1)$  and  $x_2 = \omega(A_2)$ . Also clearly  $A_1 \cup A_2 \approx A_2 \cup A_1$  and hence

$$\begin{aligned} x_1 \oplus x_2 &= \omega(A_1) \oplus \omega(A_2) \\ &= \omega(A_1 \cup A_2) = \omega(A_2 \cup A_1) = \omega(A_2) \oplus \omega(A_1) = x_2 \oplus x_1 . \end{aligned}$$

Moreover, if  $x \in X$  and  $A$  is a finite set with  $x = \omega(A)$  then

$$x \oplus x_0 = \omega(A) \oplus \omega(\emptyset) = \omega(A \cup \emptyset) = \omega(A) = x ,$$

and so  $x \oplus x_0 = x$  for all  $x \in X$ .

Let  $x_1, x_2 \in X$ , and so by Lemma 3.5 there exist finite sets  $A$  and  $B$  such that  $x_1 = \omega(A)$  and  $x_2 = \omega(B)$ . By Theorem 2.4 there either exists an injective mapping  $g : A \rightarrow B$  or an injective mapping  $h : B \rightarrow A$ . Assume the former holds and put  $B' = g(A)$  and  $C = B \setminus B'$ . Then  $B'$  and  $C$  are disjoint and  $B = B' \cup C$ ; moreover,  $A \approx B'$  (since  $g$  considered as a mapping from  $A$  to  $B'$  is a bijection) and so by Theorem 3.1 (2)  $\omega(A) = \omega(B')$ . Thus, putting  $x = \omega(C)$ , it follows that  $x_2 = \omega(B) = \omega(B' \cup C) = \omega(B') \oplus \omega(C) = \omega(A) \oplus \omega(C) = x_1 \oplus x$ . On the other hand, if there exists an injective mapping  $h : B \rightarrow A$  then the same argument shows that  $x_1 = x_2 \oplus x$  for some  $x \in X$ .

Now to (a0) and (a1), and we have seen above that (a0) holds. Let  $x, x' \in X$ , so by Lemma 6.1 there exist disjoint finite sets  $A$  and  $B$  with  $x = \omega(A)$  and  $x' = \omega(B)$ . Let  $b \notin A \cup B$ ; then

$$\begin{aligned} x \oplus f(x') &= \omega(A) \oplus f(\omega(B)) = \omega(A) \oplus \omega(B \cup \{b\}) = \omega(A \cup (B \cup \{b\})) \\ &= \omega((A \cup B) \cup \{b\}) = f(\omega(A \cup B)) = f(\omega(A) \oplus \omega(B)) = f(x \oplus x') \end{aligned}$$

and hence (a1) holds. If  $\oplus'$  is another binary operation on  $X$  satisfying (a0) and (a1) then it is easy to see that  $X_0 = \{x' \in X : x \oplus' x' = x \oplus x' \text{ for all } x \in X\}$  is an  $f$ -invariant subclass of  $X$  containing  $x_0$ . Hence  $X_0 = X$ , since  $\mathbb{I}$  is minimal, which implies that  $\oplus' = \oplus$ .

This completes the proof of Theorem 6.1.  $\square$

Theorem 6.2 will be dealt with in a similar manner. We obtain the operation  $\otimes$  by showing that  $\gamma_{\otimes}$  satisfies condition ( $\heartsuit$ ).

As with the addition  $\oplus$ , once it is known that the operation  $\otimes$  exists then the remaining properties of  $\otimes$  listed in Theorem 6.2 follow from the corresponding properties of the cartesian product operation  $\times$  (for example, that it is (modulo the relation  $\approx$ ) associative and commutative) and from the relationship between  $\cup$  and  $\times$ .

The following shows that  $\gamma_{\otimes}$  satisfies condition ( $\heartsuit$ ).

**Lemma 6.3** *If  $A, B, A', B'$  are finite sets with  $\omega(A) = \omega(A')$  and  $\omega(B) = \omega(B')$  then  $\omega(A \times B) = \omega(A' \times B')$ .*

*Proof* Consider finite sets  $A$  and  $A'$  with  $\omega(A) = \omega(A')$  and let  $B$  be any finite set. Put  $\mathcal{S} = \{C \in \mathcal{P}(B) : \omega(A \times C) = \omega(A' \times C)\}$ . Then  $\emptyset \in \mathcal{S}$ , since  $A \times \emptyset = \emptyset = A' \times \emptyset$  and so  $\omega(A \times \emptyset) = \omega(A' \times \emptyset)$ . Let  $C \in \mathcal{S}$  and let  $b \notin B \setminus C$ . Then the sets  $A \times C$  and  $A \times \{b\}$  are disjoint and  $A \times (C \cup \{b\}) = (A \times C) \cup (A \times \{b\})$ . It follows that

$$\omega(A \times (C \cup \{b\})) = \omega((A \times C) \cup (A \times \{b\})) = \omega(A \times C) \oplus \omega(A \times \{b\})$$

and in the same way  $\omega(A' \times (C \cup \{b\})) = \omega(A' \times C) \oplus \omega(A' \times \{b\})$ . Clearly  $A \times \{b\} \approx A$  and so by Theorem 3.1 (2)  $\omega(A \times \{b\}) = \omega(A)$ , and in the same way  $\omega(A' \times \{b\}) = \omega(A')$ . Therefore

$$\begin{aligned} \omega(A \times (C \cup \{b\})) &= \omega(A \times C) \oplus \omega(A \times \{b\}) = \omega(A' \times C) \oplus \omega(A) \\ &= \omega(A' \times C) \oplus \omega(A' \times \{b\}) = \omega(A' \times (C \cup \{b\})) \end{aligned}$$

and so  $B \cup \{b\} \in \mathcal{S}$ . Hence  $\mathcal{S}$  is an inductive  $B$ -system and so  $B \in \mathcal{S}$ . Therefore  $\omega(A \times B) = \omega(A' \times B)$ . Now let  $A, B, A', B'$  be finite sets with  $\omega(A) = \omega(A')$  and  $\omega(B) = \omega(B')$ . Then clearly we have  $A' \times B \approx B \times A'$  and  $A' \times B' \approx B' \times A'$  and hence by Theorem 3.1 (2)  $\omega(A' \times B) = \omega(B \times A')$  and  $\omega(A' \times B') = \omega(B' \times A')$ . Hence by the first part

$$\omega(A \times B) = \omega(A' \times B) = \omega(B \times A') = \omega(B' \times A') = \omega(A' \times B'). \quad \square$$



*Proof of Theorem 6.2:* By Lemma 6.3 condition  $(\heartsuit)$  holds for the mapping  $\gamma_\otimes$  and thus by Proposition 6.1 there exists a binary operation  $\otimes$  on  $X$  such that  $\omega(A) \otimes \omega(B) = \omega(A \times B)$  whenever  $A$  and  $B$  are finite sets. Moreover,  $\otimes$  is the unique operation with this property.

We show that  $\otimes$  is associative and commutative: Let  $x_1, x_2, x_3 \in X$ ; then by Lemma 3.5 there exists finite sets  $A_1, A_2, A_3$  with  $x_1 = \omega(A_1)$ ,  $x_2 = \omega(A_2)$  and  $x_3 = \omega(A_3)$ . Now it is easy to check that  $(A_1 \times A_2) \times A_3 \approx A_1 \times (A_2 \times A_3)$  and so by Theorem 3.1 (2)  $\omega((A_1 \times A_2) \times A_3) = \omega(A_1 \times (A_2 \times A_3))$ . Therefore

$$\begin{aligned} (x_1 \otimes x_2) \otimes x_3 &= (\omega(A_1) \otimes \omega(A_2)) \otimes \omega(A_3) = \omega(A_1 \times A_2) \otimes \omega(A_3) \\ &= \omega((A_1 \times A_2) \times A_3) = \omega(A_1 \times (A_2 \times A_3)) \\ &= \omega(A_1) \otimes \omega(A_2 \times A_3) = \omega(A_1) \otimes (\omega(A_2) \otimes \omega(A_3)) \\ &= x_1 \otimes (x_2 \otimes x_3) \end{aligned}$$

which shows  $\otimes$  is associative. Let  $x_1, x_2 \in X$ ; by Lemma 3.5 there exist finite sets  $A_1$  and  $A_2$  with  $x_1 = \omega(A_1)$  and  $x_2 = \omega(A_2)$ . Then by Theorem 3.1 (2) we have  $\omega(A_1 \times A_2) = \omega(A_2 \times A_1)$ , since clearly  $A_1 \times A_2 \approx A_2 \times A_1$ . Thus

$$\begin{aligned} x_1 \otimes x_2 &= \omega(A_1) \otimes \omega(A_2) \\ &= \omega(A_1 \times A_2) = \omega(A_2 \times A_1) = \omega(A_2) \otimes \omega(A_1) = x_2 \otimes x_1 \end{aligned}$$

which shows that  $\otimes$  is also commutative.

Let  $x \in X$ , so by Lemma 3.5 there exists a finite set  $A$  with  $x = \omega(A)$ . Then

$$x \otimes x_0 = \omega(A) \otimes \omega(\emptyset) = \omega(A \times \emptyset) = \omega(\emptyset) = x_0 .$$

Moreover, if  $x \neq x_0$  then  $A \neq \emptyset$ , so if  $a$  is any element then by Theorem 3.1 (2)  $\omega(A \times \{a\}) = \omega(A)$ , since  $A \times \{a\} \approx A$ , and hence

$$\begin{aligned} x \otimes f(x_0) &= \omega(A) \otimes f(\omega(\emptyset)) = \omega(A) \otimes \omega(\emptyset \cup \{a\}) \\ &= \omega(A) \otimes \omega(\{a\}) = \omega(A \times \{a\}) = \omega(A) = x . \end{aligned}$$

Thus  $x \otimes x_0 = x_0$  for each  $x \in X$  and  $x \otimes f(x_0) = x$  for each  $x \neq x_0$  (and note that the first statement is  $(m0)$ ).

Now for the distributive law. Let  $x, x_1, x_2 \in X$ . There exists a finite set  $A$  with  $x = \omega(A)$  and disjoint finite sets  $B$  and  $C$  with  $x_1 = \omega(B)$  and  $x_2 = \omega(C)$ . Then  $A \times (B \cup C)$  is the disjoint union of  $A \times B$  and  $A \times C$  and thus

$$\begin{aligned} (x \otimes x_1) \oplus (x \otimes x_2) &= (\omega(A) \otimes \omega(B)) \oplus (\omega(A) \otimes \omega(C)) \\ &= \omega(A \times B) \oplus \omega(A \times C) = \omega((A \times B) \cup (A \times C)) \\ &= \omega(A \times (B \cup C)) = \omega(A) \otimes \omega(B \cup C) \\ &= \omega(A) \otimes (\omega(B) \oplus \omega(C)) = x \otimes (x_1 \oplus x_2) . \end{aligned}$$

We have already seen that (m0) holds and, since  $f(x_0)$  is an identity element, (m1) is a special case of the distributive law: Let  $x, x' \in X$ ; then by (a0) and (a1) and since  $\oplus$  is commutative it follows that  $f(x') = f(x' \oplus x_0) = x' \oplus f(x_0) = f(x_0) \oplus x'$ , and hence  $x \otimes f(x') = x \otimes (f(x_0) \oplus x') = (x \otimes f(x_0)) \oplus (x \otimes x') = x \oplus (x \otimes x')$ , which is (m1). Finally, if  $\otimes'$  is another binary operation satisfying (m0) and (m1) then it is easy to see that  $X_0 = \{x' \in X : x \otimes' x' = x \otimes x' \text{ for all } x \in X\}$  is a  $f$ -invariant subclass of  $X$  containing  $x_0$ . Hence  $X_0 = X$ , since  $\mathbb{I}$  is minimal, which implies that  $\otimes' = \otimes$ . This completes the proof of Theorem 6.2.  $\square$

We next give some results about the operation  $\oplus$  for special cases of  $\mathbb{I}$ .

**Proposition 6.2** *If  $f$  is injective then the cancellation law holds for  $\oplus$  (meaning that  $x_1 = x_2$  whenever  $x_1 \oplus x = x_2 \oplus x$  for some  $x \in X$ ). In particular,  $x \neq x \oplus x'$  for all  $x, x' \in X$  with  $x' \neq x_0$  (since  $x = x \oplus x_0$ ).*

*Proof* Let  $x_1, x_2 \in X$  with  $x_1 \neq x_2$  and let  $X_0 = \{x \in X : x_1 \oplus x \neq x_2 \oplus x\}$ ; then  $x_0 \in X_0$ , since by (a0)  $x_1 \oplus x_0 = x_1 \neq x_2 = x_2 \oplus x_0$ . Let  $x \in X_0$ , then by (a1), and since  $f$  is injective,  $x_1 \oplus f(x) = f(x_1 \oplus x) \neq f(x_2 \oplus x) = x_2 \oplus f(x)$ , i.e.,  $f(x) \in X_0$ . Thus  $X_0$  is a  $f$ -invariant subset of  $X$  containing  $x_0$  and so  $X_0 = X$ , since  $\mathbb{I}$  is minimal. Hence if  $x_1 \neq x_2$  then  $x_1 \oplus x \neq x_2 \oplus x$  for all  $x \in X$ , which shows that the cancellation law holds for  $X$ .  $\square$

**Proposition 6.3** *If  $x_0 \in f(X)$  (and so by Theorem 3.4 and Proposition 3.4  $X$  is finite and  $f$  is bijective) then  $(X, \oplus, x_0)$  is an abelian group: For each  $x \in X$  there exists  $x' \in X$  such that  $x \oplus x' = x_0$ . Moreover,  $X$  is the group generated by the element  $f(x_0)$ .*

*Proof* By Lemma 3.5 there exists a non-empty finite set  $A$  with  $\omega(A) = x_0$  and then for each finite set  $C$  there exists  $B \subset A$  with  $\omega(B) = \omega(C)$ . By Lemma 3.5 there exists a finite set  $C$  with  $x = \omega(C)$  and hence there also exists a finite set  $B \subset A$  with  $x = \omega(B)$ . Put  $B' = A \setminus B$  and let  $x' = \omega(B')$ . Then  $B$  and  $B'$  are disjoint and hence  $x \oplus x' = \omega(B) \oplus \omega(B') = \omega(B \cup B') = \omega(A) = x_0$ . Let  $X_0$  be the least subgroup of  $X$  containing  $f(x_0)$ . Then  $x_0 \in X_0$  and if  $x \in X_0$  then by (a0) and (a1)  $f(x) = f(x \oplus x_0) = x \oplus f(x_0)$  and hence  $f(x) \in X_0$ . Thus  $X_0$  is a  $f$ -invariant subset of  $X$  containing  $f(x_0)$  and so  $X_0 = X$ . Therefore  $X$  is the group generated by  $f(x_0)$ .  $\square$

Until further notice let  $\mathbb{I}$  be a Peano iterator. We will introduce the order relation  $\leq$  corresponding to that defined on the natural numbers.

**Lemma 6.4** *If  $x_1, x_2 \in X$  then  $x_1 \oplus x_2 = x_0$  if and only if  $x_1 = x_2 = x_0$ .*

*Proof* Suppose that  $x_1 \oplus x_2 = x_0$ . Now there exist disjoint finite sets  $A_1, A_2$  with  $\omega(A_1) = x_1$  and  $\omega(A_2) = x_2$  and then  $x_1 \oplus x_2 = \omega(A_1 \cup A_2) = \omega(\emptyset)$ . Therefore  $A_1 \cup A_2 \approx \emptyset$ , since  $\mathbb{I}$  is a Peano iterator. Thus  $A_1 = A_2 = \emptyset$ , i.e.,  $x_1 = x_2 = x_0$ . The converse holds trivially  $\square$

**Proposition 6.4** *For all  $x_1, x_2 \in X$  with  $x_1 \neq x_2$  exactly one of the following two statements holds:*

*There exists a unique  $x \in X$  such that  $x_1 = x_2 \oplus x$ .*

*There exists a unique  $x' \in X$  such that  $x_2 = x_1 \oplus x'$ .*

*Proof* Note that if there exists  $x \in X$  with  $x_1 = x_2 \oplus x$  then by Proposition 6.2  $x$  is unique and in the same way if there exists  $x' \in X$  with  $x_2 = x_1 \oplus x'$  then  $x'$  is unique. Also Theorem 6.1 states that at least one of the statements holds. Suppose there exists both  $x \in X$  with  $x_1 = x_2 \oplus x$  and  $x' \in X$  with  $x_2 = x_1 \oplus x'$ . Then  $x_1 = x_2 \oplus x = (x_1 \oplus x') \oplus x = x_1 \oplus (x' \oplus x)$ . Thus by Proposition 6.2  $x' \oplus x = x_0$  and by Lemma 6.4 it then follows that  $x = x' = x'$ . But this implies  $x_1 = x_2$ , contradicting the assumption that  $x_1 \neq x_2$ . Therefore exactly one of the statements holds.  $\square$

Now define a binary relation  $\leq$  on  $X$  by: If  $x_1, x_2 \in X$  then  $x_1 \leq x_2$  if and only if there exists  $x \in X$  such that  $x_2 = x_1 \oplus x$ .

**Proposition 6.5** *The relation  $\leq$  is a total order on  $X$ . Moreover,  $x_1 \leq x_2$  if and only if  $f(x_1) \leq f(x_2)$ .*

*Proof* Clearly  $x \leq x$  for all  $x \in X$  since  $x = x \oplus x_0$ . Proposition 6.4 shows that for all  $x_1, x_2 \in X$  either  $x_1 \leq x_2$  or  $x_2 \leq x_1$  and if both  $x_1 \leq x_2$  and  $x_2 \leq x_1$  hold then  $x_1 = x_2$ . Finally, if  $x_1 \leq x_2$  and  $x_2 \leq x_3$  then there exist  $x, x' \in X$  with  $x_2 = x_1 \oplus x$  and  $x_3 = x_2 \oplus x'$  and thus  $x_3 = (x_1 \oplus x) \oplus x' = x_1 \oplus (x \oplus x')$ , which shows that  $x_1 \leq x_3$ . Thus  $\leq$  is a total order. Now if  $x_1 \leq x_2$  then there exists  $x \in X$  with  $x_2 = x_1 \oplus x$  and then  $f(x_2) = f(x_1 \oplus x) = f(x_1) \oplus f(x)$  and so  $f(x_1) \leq f(x_2)$ . Conversely, if  $f(x_1) \leq f(x_2)$  then there exists  $x \in X$  with  $f(x_2) = f(x_1) \oplus x = f(x_1 \oplus x)$  and hence  $x_2 = x_1 \oplus x$ , since  $f$  is injective. Thus  $x_1 \leq x_2$ .  $\square$

Recall that in Section 4 the order relation  $\leq$  on the class of finite ordinals  $O$  was defined by  $o_1 \leq o_2$  if and only if  $o_1 \subset o_2$ , and that  $\varrho(o) = o$  for all  $o \in O$ . The following result thus shows that this definition for the finite ordinals agrees with the above definition.

**Lemma 6.5** *Let  $x_1, x_2 \in X$ ; then  $x_1 \leq x_2$  if and only if there exist finite sets  $A_1, A_2$  with  $\omega(A_i) = x_i$  for  $i = 1, 2$  and  $A_1 \subset A_2$ .*

*Proof* If there exist finite sets  $A_1, A_2$  with  $\omega(A_i) = x_i$  for  $i = 1, 2$  and  $A_1 \subset A_2$  then  $x_2 = x_1 \oplus x$  with  $x = \omega(A_2 \setminus A_1)$  and so  $x_1 \leq x_2$ . Conversely, if  $x_1 \leq x_2$  then there exists  $x \in X$  with  $x_2 = x_1 \oplus x$  and if  $A, A_1$  are disjoint finite sets with  $x_1 = \omega(A_1)$  and  $x = \omega(A)$  then  $x_2 = x_1 \oplus x = \omega(A_1 \cup A)$  and  $A_1 \subset A_1 \cup A$ .  $\square$

**Proposition 6.6** *Let  $x, x' \in X$ ; then  $x' \leq f(x)$  if and only if  $x' \leq x$  or  $x' = f(x)$ .*

*Proof* Suppose first that  $x' \leq f(x)$ ; then by Lemma 6.5 there exist finite sets  $A, A'$  with  $\omega(A) = f(x)$ ,  $\omega(A') = x'$  and  $A' \subset A$ . If  $A' = A$  then  $x' = f(x)$ . If  $A' \neq A$  then  $A'$  is a proper subset of  $A$  and so choose  $a \in A \setminus A'$ . Then  $A' \subset A \setminus \{a\}$  and so  $x' \leq \omega(A \setminus \{a\})$ . But  $f(\omega(A \setminus \{a\})) = \omega(A) = x$ . Hence if  $x' \leq f(x)$  then either  $x' \leq x$  or  $x' = f(x)$ . Suppose conversely that  $x' \leq x$ ; then by Lemma 6.5 there exist finite sets  $A, A'$  with  $\omega(A) = x$ ,  $\omega(A') = x'$  and  $A' \subset A$ . Let  $a$  be an element not in  $A$ . Then  $A' \subset A \cup \{a\}$  and  $\omega(A \cup \{a\}) = f(x)$  and so  $x' \leq f(x)$ . Thus if  $x' \leq x$  or  $x' = f(x)$  then  $x' \leq f(x)$ , since if  $x' = f(x)$  then trivially  $x' \leq f(x)$ .  $\square$

**Lemma 6.6** *For each  $x \in X$  the subclass  $X_x = \{x' \in X : x' \leq x\}$  is a finite set.*

*Proof* Let  $X_0 = \{x \in X : X_x \text{ is a finite set}\}$ . Then  $x_0 \in X_0$ , since  $X_{x_0} = \{x_0\}$  and by Proposition 6.6  $X_0$  is  $f$ -invariant. Thus  $X_0 = X$ , since  $\mathbb{I}$  is minimal.  $\square$

A subclass  $X_0$  of  $X$  is said to be bounded if there exists  $x \in X$  such that  $x' \leq x$  for all  $x' \in X_0$ . By Lemma 6.6 a bounded subclass of  $X$  is a finite set.

**Proposition 6.7** (1) *Each non-empty subclass of  $X$  contains a minimum element.*

(2) *Each non-empty bounded subclass of  $X$  contains a maximum element.*

*Proof* (1) Let  $X_0$  be a non-empty subclass of  $X$  and let  $x \in X_0$ . Then Lemma 6.6 implies that  $X_1 = \{x' \in X_0 : x' \leq x\}$  is a non-empty finite set and thus by Proposition 2.14 it contains a minimum element which is then the minimum element of  $X_0$ .

(2) This follows immediately from Proposition 2.14 and Lemma 6.6.  $\square$

We end the section by looking at the operation of exponentiation. Here we have to be more careful: For example,  $2 \cdot 2 \cdot 2 = 2$  in  $\mathbb{Z}_3$  and so  $2^3$  is not well-defined if the exponent 3 is considered as an element of  $\mathbb{Z}_3$  (since we would also have to have  $2^0 = 1$ ). However,  $2^3$  does make sense if 2 is considered as an element of  $\mathbb{Z}_3$  and the exponent 3 as an element of  $\mathbb{N}$ .

In general we will see that if  $(Y, g, y_0)$  is a Peano iterator then we can define an element of  $X$  which is ‘ $x$  to the power of  $y$ ’ for each  $x \in X$  and each  $y \in Y$  and this operation has the properties which might be expected.

In what follows let  $\mathbb{J} = (Y, g, y_0)$  be a Peano iterator with  $\omega'$  the assignment of finite sets in  $\mathbb{J}$ . (As before  $\mathbb{I} = (X, f, x_0)$  is assumed to be minimal with  $\omega$  the assignment of finite sets in  $\mathbb{I}$ .) Also let  $\oplus$  and  $\otimes$  be the operations given in Theorems 6.1 and 6.2 for the iterator  $\mathbb{I}$ .

**Theorem 6.3** *There exists a unique operation  $\uparrow : X \times Y \rightarrow X$  such that*

$$\omega(A) \uparrow \omega'(B) = \omega(A^B)$$

*for all finite sets  $A$  and  $B$ . This operation  $\uparrow$  satisfies*

$$x \uparrow (y_1 \oplus y_2) = (x \uparrow y_1) \otimes (x \uparrow y_2)$$

*for all  $x \in X$  and all  $y_1, y_2 \in Y$  and*

$$(x_1 \otimes x_2) \uparrow y = (x_1 \uparrow y) \otimes (x_2 \uparrow y)$$

*for all  $x_1, x_2 \in X$  and  $y \in Y$ . Moreover,  $\uparrow$  is the unique operation such that*

$$(e0) \quad x \uparrow y_0 = f(x_0) \text{ for all } x \in X.$$

$$(e1) \quad x \uparrow g(y) = x \otimes (x \uparrow y) \text{ for all } x \in X, y \in Y.$$

**Lemma 6.7** *If  $B, C$  are finite sets with  $\omega(B) = \omega(C)$  then for all finite sets  $A$  we have  $\omega(B^A) = \omega(C^A)$ .*

*Proof* Let  $B$  and  $C$  be finite sets with  $\omega(B) = \omega(C)$ , let  $A$  be a finite set and put  $\mathcal{S} = \{D \in \mathcal{P}(A) : \omega(B^D) = \omega(C^D)\}$ . Then  $\emptyset \in \mathcal{S}$ , since  $\omega(B^\emptyset) = \omega(C^\emptyset)$ . (For any set  $X$  the set  $X^\emptyset$  consists of the single element  $\{\emptyset\}$ .) Let  $D \in \mathcal{S}^p$  and  $a \in A \setminus D$ . Now  $\omega(B^D) = \omega(C^D)$  (since  $D \in \mathcal{S}$ ) and  $\omega(B) = \omega(C)$ ; therefore by Lemma 6.3 and Theorem 3.1 (2)

$$\omega(B^{D \cup \{a\}}) = \omega(B^D \times B) = \omega(C^D \times C) = \omega(C^{D \cup \{a\}})$$

(since  $E^{D \cup \{a\}} \approx E^D \times E$  for each set  $E$ ), and so  $D \cup \{a\} \in \mathcal{S}$ . Hence  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Thus  $\omega(B^A) = \omega(C^A)$ .  $\square$

*Remark:* If  $B, C$  are finite sets with  $\omega(B) = \omega(C)$  then  $\omega(A^B) = \omega(A^C)$  does not hold in general for a finite set  $A$ .

*Proof of Theorem 6.3:* Let  $A_1, A_2, B_1, B_2$  be finite sets with  $\omega(A_1) = \omega(A_2)$  and  $\omega'(B_1) = \omega'(B_2)$ ; then by Lemma 6.7  $\omega(A_1^{B_1}) = \omega(A_2^{B_1})$  and by Theorem 3.2  $B_1 \approx B_2$ . Since  $B_1 \approx B_2$  it follows that  $A_2^{B_1} \approx A_2^{B_2}$  and then by Theorem 3.1 (2)  $\omega(A_2^{B_1}) = \omega(A_2^{B_2})$ . This shows that  $\omega(A_1^{B_1}) = \omega(A_2^{B_2})$ . Therefore by Lemma 3.5 we can define  $x \uparrow y$  to be  $\omega(A^B)$ , where  $A$  and  $B$  are any finite sets with  $x = \omega(A)$  and  $y = \omega'(B)$ . Then  $\omega(A) \uparrow \omega'(B) = \omega(A^B)$  for all finite sets  $A$  and  $B$  and this requirement clearly determines  $\uparrow$  uniquely.

Let  $x \in X$  and  $y_1, y_2 \in Y$ ; then by Lemma 6.1 there exists a disjoint pair  $(B_1, B_2)$  with  $(y_1, y_2) = \omega'(B_1, B_2)$  and by Lemma 3.5 there exists a finite set  $A$  with  $x = \omega(A)$ . Moreover, it is easily checked that  $A^{B_1 \cup B_2} \approx A^{B_1} \times A^{B_2}$  and thus by Theorem 3.1 (2)

$$\begin{aligned} x \uparrow (y_1 \oplus y_2) &= \omega(A) \uparrow (\omega'(B_1) \oplus \omega'(B_2)) \\ &= \omega(A) \uparrow \omega'(B_1 \cup B_2) = \omega(A^{B_1 \cup B_2}) = \omega(A^{B_1} \times A^{B_2}) \\ &= \omega(A^{B_1}) \otimes \omega(A^{B_2}) = (x \uparrow y_1) \otimes (x \uparrow y_2). \end{aligned}$$

Now let  $x_1, x_2 \in X$  and  $y \in Y$ . By Lemma 3.5 there exist finite sets  $A_1, A_2$  and  $B$  such that  $x_1 = \omega(A_1)$ ,  $x_2 = \omega(A_2)$  and  $y = \omega'(B)$  and  $(A_1 \times A_2)^B \approx A_1^B \times A_2^B$ . Thus by Theorem 3.1 (2)

$$\begin{aligned} (x_1 \otimes x_2) \uparrow y &= (\omega(A_1) \otimes \omega(A_2)) \uparrow \omega'(B) \\ &= \omega(A_1 \times A_2) \uparrow \omega'(B) = \omega((A_1 \times A_2)^B) = \omega(A_1^B \times A_2^B) \\ &= \omega(A_1^B) \otimes \omega(A_2^B) = (x_1 \uparrow y) \otimes (x_2 \uparrow y). \end{aligned}$$

It remains to consider the properties (e0) and (e1). Now for each finite set  $A$  we have  $\omega(A) \uparrow \omega'(\emptyset) = \omega(A^\emptyset) = \omega(\{\emptyset\}) = f(x_0)$  and hence  $x \uparrow y_0 = f(x_0)$  for each  $x \in X$ , i.e., (e0) holds. Let  $A$  and  $B$  be finite sets and let  $b \notin B$ . Then, since  $A^{B \cup \{b\}} \approx A \times A^B$ , it follows from Theorem 3.1 (2) that

$$\begin{aligned} \omega(A) \uparrow g(\omega'(B)) &= \omega(A) \uparrow \omega'(B \cup \{b\}) = \omega(A^{B \cup \{b\}}) = \omega(A \times A^B) \\ &= \omega(A) \otimes \omega(A^B) = \omega(A) \otimes (\omega(A) \uparrow \omega'(B)) \end{aligned}$$

and this shows  $x \uparrow g(h) = x \otimes (x \uparrow y)$  for all  $x \in X$ ,  $y \in Y$ , i.e., (e1) holds. Finally, if  $\uparrow'$  is another operation satisfying (e0) and (e1) then

$$Y_0 = \{y \in Y : x \uparrow' y = x \uparrow y \text{ for all } x \in X\}$$

is a  $g$ -invariant subset of  $Y$  containing  $y_0$ . Therefore  $Y_0 = Y$ , since  $(Y, \cdot, y_0)$  is minimal, which implies that  $\uparrow' = \uparrow$ .  $\square$

## 7 Another take on addition and multiplication

In the following again let  $\mathbb{I} = (X, f, x_0)$  be a minimal iterator with  $\omega$  the assignment of finite sets in  $\mathbb{I}$ . In this section we give alternative proofs for Theorems 6.1 and 6.2.

In Section 6 only the single assignment  $\omega$  was used. Here we make use of a family of assignments  $\{\omega_x : x \in X\}$ , which arise as follows: For each  $x \in X$  there is the iterator  $\mathbb{I}_x = (X, f, x)$  (which will usually not be minimal) and the assignment of finite sets in  $\mathbb{I}_x$  will be denoted by  $\omega_x$ . Thus  $\omega_x(\emptyset) = x$  and  $\omega_x(A \cup \{a\}) = f(\omega_x(A))$  whenever  $A$  is a finite set and  $a \notin A$ . In particular we have  $\omega = \omega_{x_0}$ . Now it is more convenient to repackage the information given by the assignments  $\omega_A$ ,  $x \in X$ , by introducing for each finite set  $A$  the mapping  $f_A : X \rightarrow X$  with  $f_A(x) = \omega_x(A)$  for all  $x \in X$ , and so  $\omega(A) = \omega_{x_0}(A) = f_A(x_0)$ .

Consider disjoint finite sets  $A$  and  $B$ ; then  $\omega(A \cup B)$  can be thought of as the element of  $X$  obtained by iterating  $f$  through the elements of  $A \cup B$  starting with  $x_0$ . This element can also be determined by first iterating  $f$  through the elements of  $B$  starting with  $x_0$ , giving the result  $\omega(B)$  and then iterating  $f$  through the elements of  $A$ , but starting with the element  $\omega(B)$  and not with  $x_0$ . The result is  $\omega_z(A)$ , where  $z = \omega(B)$ , and  $\omega_z(A) = f_A(z) = f_A(\omega(B)) = f_A(f_B(x_0)) = (f_A \circ f_B)(x_0)$ , and so we would expect that  $\omega(A \cup B) = (f_A \circ f_B)(x_0)$ . But if  $\oplus$  is the operation given by Theorem 6.1 then  $\omega(A \cup B) = \omega(A) \oplus \omega(B) = f_A(x_0) \oplus f_B(x_0)$ , which suggests that the following should hold:

( $\diamond$ )  $f_A(x_0) \oplus f_B(x_0) = (f_A \circ f_B)(x_0)$  whenever  $A$  and  $B$  are disjoint finite sets.

It will be seen later that ( $\diamond$ ) does hold. What is perhaps more important, though, is that ( $\diamond$ ) can actually be used to define  $\oplus$ , as we now explain.

Denote by  $T_X$  the set of all mappings from  $X$  to itself and so  $A \mapsto f_A$  defines a mapping from  $\text{Fin}$  to  $T_X$ . Then  $(T_X, \circ, \text{id}_X)$ , where  $\circ$  is functional composition and  $\text{id}_X : X \rightarrow X$  is the identity mapping, is a monoid. (A monoid is any triple  $(M, \bullet, e)$  consisting of a class  $M$ , an associative operation  $\bullet$  on  $M$  and an identity element  $e \in M$  satisfying  $a \bullet e = e \bullet a = a$  for all  $a \in M$ .) Lemma 7.6 shows that

$$M_f = \{u \in T_X : u = f_A \text{ for some finite set } A\}$$

is a submonoid of  $(T_X, \circ, \text{id}_X)$ , meaning that  $\text{id}_X \in M_f$  and  $u_1 \circ u_2 \in M_f$  for all  $u_1, u_2 \in M_f$ , and that this submonoid is commutative, i.e.,  $u_1 \circ u_2 = u_2 \circ u_1$  for all  $u_1, u_2 \in M_f$ . (The monoid  $(T_X, \circ, \text{id}_X)$  itself is not commutative except when  $X = \{x_0\}$ .)

Let  $\Phi_{x_0} : M_f \rightarrow X$  be the mapping with  $\Phi_{x_0}(u) = u(x_0)$  for each  $u \in M_f$ , and so in particular  $\Phi_{x_0}(f_A) = f_A(x_0) = \omega(A)$  for each finite set  $A$ . Lemma 7.7 will show

that  $\Phi_{x_0}$  is a bijection, and therefore there exists a unique operation  $\oplus$  on  $X$  such that

$$(\heartsuit) \quad \Phi_{x_0}(u) \oplus \Phi_{x_0}(v) = \Phi_{x_0}(u \circ v) \text{ for all } u, v \in M_f.$$

This is how  $\oplus$  will be defined below. Note that if  $A$  and  $B$  are (not necessarily disjoint) finite sets then by  $(\heartsuit)$

$$f_A(x_0) \oplus f_B(x_0) = \Phi_{x_0}(f_A) \oplus \Phi_{x_0}(f_B) = \Phi_{x_0}(f_A \circ f_B) = (f_A \circ f_B)(x_0)$$

and so in particular  $(\diamond)$  holds.

We now give the details of the approach outlined above.

**Lemma 7.1** *The mapping  $A \mapsto f_A$  is the unique mapping from  $\text{Fin}$  to  $T_X$  with  $f_\emptyset = \text{id}_X$  such that  $f_{A \cup \{a\}} = f \circ f_A$  whenever  $A$  is a finite set and  $a \notin A$ .*

*Proof* We have  $f_\emptyset(x) = \omega_x(\emptyset) = x = \text{id}_X(x)$  for all  $x \in X$ , and thus  $f_\emptyset = \text{id}_X$ . Moreover, if  $A$  is a finite set and  $a \notin A$  then

$$f_{A \cup \{a\}}(x) = \omega_x(A \cup \{a\}) = f(\omega_x(A)) = f(f_A(x)) = (f \circ f_A)(x)$$

for all  $x \in X$  and hence  $f_{A \cup \{a\}} = f \circ f_A$ . Finally, consider a further mapping  $A \mapsto f'_A$  with  $f'_\emptyset = \text{id}_X$  and such that  $f'_{A \cup \{a\}} = f \circ f'_A$  whenever  $A$  is a finite set and  $a \notin A$ . Let  $A$  be a finite set and put  $\mathcal{S} = \{B \in \mathcal{P}(A) : f'_B = f_B\}$ . Then  $\emptyset \in \mathcal{S}$ , since  $f'_\emptyset = \text{id}_X = f_\emptyset$ . Let  $B \in \mathcal{S}^p$  (and so  $f'_B = f_B$ ) and let  $a \in A \setminus B$ . Then  $f'_{B \cup \{a\}} = f \circ f'_B = f \circ f_B = f_{B \cup \{a\}}$  and therefore  $B \cup \{a\} \in \mathcal{S}$ . Thus  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Hence  $f'_A = f_A$ .  $\square$

The mapping  $A \mapsto f_A$  will be called the functional assignment of finite sets in  $\mathbb{I}$ . In particular  $f_{\{a\}} = f$  for each element  $a$ , since  $f_{\{a\}} = f_{\emptyset \cup \{a\}} = f \circ f_\emptyset = f \circ \text{id}_X = f$ .

**Lemma 7.2**  *$f \circ f_A = f_A \circ f$  for each finite set  $A$ .*

*Proof* Let  $A$  be a finite set and put  $\mathcal{S} = \{B \in \mathcal{P}(A) : f \circ f_B = f_B \circ f\}$ . Then  $\emptyset \in \mathcal{S}$  since  $f \circ f_\emptyset = f \circ \text{id}_X = f = \text{id}_X \circ f = f_\emptyset \circ f$ . Let  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ . Then  $f \circ f_{B \cup \{a\}} = f \circ f \circ f_B = f \circ f_B \circ f = f_{B \cup \{a\}} \circ f$  and so  $B \cup \{a\} \in \mathcal{S}$ . Thus  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Hence  $f \circ f_A = f_A \circ f$ .  $\square$

The next result establishes an important relationship between  $\omega$  and the functional assignment.



**Proposition 7.1** *If  $A$  and  $B$  are finite sets then  $f_A = f_B$  holds if and only if  $\omega(A) = \omega(B)$ .*

*Proof* By definition  $\omega(C) = f_C(x_0)$  for each finite set  $C$ , and so  $\omega(A) = \omega(B)$  whenever  $f_A = f_B$ . Suppose conversely that  $\omega(A) = \omega(B)$  and consider the set  $X_0 = \{x \in X : f_A(x) = f_B(x)\}$ . Then  $X_0$  is  $f$ -invariant, since if  $x \in X_0$  then by Lemma 7.2  $f_A(f(x)) = f(f_A(x)) = f(f_B(x)) = f_B(f(x))$ , i.e.,  $f(x) \in X_0$ . Also  $x_0 \in X_0$ , since  $f_A(x_0) = \omega(A) = \omega(B) = f_B(x_0)$ . Hence  $X_0 = X$ , since  $\mathbb{I}$  is minimal. This shows that  $f_A(x) = f_B(x)$  for all  $x \in X$ , i.e.,  $f_A = f_B$ .  $\square$

There is another way of obtaining the functional assignment: Consider the iterator  $\mathbb{I}_* = (\mathbb{T}_X, f_*, \text{id}_X)$ , where  $f_* : \mathbb{T}_X \rightarrow \mathbb{T}_X$  is defined by  $f_*(h) = f \circ v$  for all  $v \in \mathbb{T}_X$ .

**Lemma 7.3** *If  $\omega_*$  is the assignment of finite sets in  $\mathbb{I}_*$  then  $\omega_*(A) = f_A$  for each finite set  $A$ .*

*Proof* By definition  $\omega_*(\emptyset) = \text{id}_X$  and  $\omega_*(A \cup \{a\}) = f_*(\omega_*(A)) = f \circ \omega_*(A)$  for each finite set  $A$  and each  $a \notin A$ . Thus by the uniqueness in Lemma 7.1  $\omega_*(A) = f_A$  for each finite set  $A$ .  $\square$

**Proposition 7.2** *If  $A$  and  $B$  are finite sets with  $A \approx B$  then  $f_A = f_B$ .*

*Proof* This follows from Theorem 3.1 (applied to  $\omega_*$ ) and Lemma 7.3.  $\square$

**Lemma 7.4** *If  $A$  and  $B$  are disjoint finite sets then  $f_{A \cup B} = f_A \circ f_B$ .*

*Proof* Let  $A$  and  $B$  be finite sets and let  $\mathcal{S} = \{C \in \mathcal{P}(A) : f_{C \cup B} = f_C \circ f_B\}$ . Then  $\emptyset \in \mathcal{S}$  since  $f_{\emptyset \cup B} = f_B = \text{id}_X \circ f_B = f_{\emptyset} \circ f_B$ . Let  $C \in \mathcal{S}$  and let  $a \in A \setminus C$ ; then  $f_{C \cup B} = f_C \circ f_B$  and hence by Lemma 7.1

$$f_{(C \cup \{a\}) \cup B} = f_{(C \cup B) \cup \{a\}} = f \circ f_{C \cup B} = f \circ f_C \circ f_B = f_{C \cup \{a\}} \circ f_B.$$

This shows that  $C \cup \{a\} \in \mathcal{S}$ . Thus  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Hence  $f_{A \cup B} = f_A \circ f_B$ .  $\square$

**Lemma 7.5** (1) *If  $f$  is bijective then  $f_A$  is bijective for each finite set  $A$ .*

(2) *If  $f$  is injective then  $f_A$  is also injective for each finite set  $A$ .*

*Proof (1)* Let  $A$  be a finite set and put  $\mathcal{S} = \{B \in \mathcal{P}(A) : f_B \text{ is bijective}\}$ . Then  $\emptyset \in \mathcal{S}$ , since  $f_\emptyset = \text{id}_X$  is bijective. Consider  $B \in \mathcal{S}^p$  and let  $a \in A \setminus B$ . Then  $f_{B \cup \{a\}} = f \circ f_B$  and so  $f_{B \cup \{a\}}$ , as the composition of two bijective mappings, is itself bijective, i.e.,  $B \cup \{a\} \in \mathcal{S}$ . Thus  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Hence  $f_A$  is bijective.

(2) Just replace ‘bijective’ by ‘injective’ in (1).  $\square$

As above let  $M_f = \{u \in T_X : u = f_A \text{ for some finite set } A\}$ . Then in particular  $\text{id}_X \in M_f$ , since  $\text{id}_X = f_\emptyset$ , and  $f \in M_f$ , since  $f = f_{\{a\}}$  for each element  $a$ . Moreover, if  $f$  is injective (resp. bijective) then by Lemma 7.5 each element in  $M_f$  is injective (resp. bijective).

**Lemma 7.6** For all  $u_1, u_2 \in M_f$  we have  $u_1 \circ u_2 \in M_f$  and  $u_1 \circ u_2 = u_2 \circ u_1$ . (Since also  $\text{id}_X \in M_f$  this means that  $M_f$  is a commutative submonoid of the monoid  $(T_X, \circ, \text{id}_X)$ .)

*Proof* Let  $u_1, u_2 \in M_f$  and so there exist finite sets  $A$  and  $B$  with  $u_1 = f_A$  and  $u_2 = f_B$ . There then exists a disjoint pair  $(A', B')$  with  $(A', B') \approx (A, B)$  and hence by Proposition 7.2 and Lemma 7.4

$$u_1 \circ u_2 = f_A \circ f_B = f_{A'} \circ f_{B'} = f_{A' \cup B'} = f_{B' \cup A'} = f_{B'} \circ f_{A'} = f_B \circ f_A = u_2 \circ u_1 ,$$

i.e.,  $u_1 \circ u_2 = u_2 \circ u_1$ . Moreover, since  $u_1 \circ u_2 = f_{A' \cup B'}$  and  $f_{A' \cup B'} \in M_f$ , this also shows that  $u_1 \circ u_2 \in M_f$ .  $\square$

As above let  $\Phi_{x_0} : M_f \rightarrow X$  be the mapping with  $\Phi_{x_0}(u) = u(x_0)$  for all  $u \in M_f$ . Then  $\Phi_{x_0}(\text{id}_X) = x_0$  and  $\Phi_{x_0}(f_A) = f_A(x_0) = \omega(A)$  for each finite set  $A$ . An important property of  $\Phi_{x_0}$  is that

$$(\#) \quad u(\Phi_{x_0}(v)) = \Phi_{x_0}(u \circ v) \text{ for all } u, v \in M_f,$$

which holds since  $u(\Phi_{x_0}(v)) = u(v(x_0)) = (u \circ v)(x_0) = \Phi_{x_0}(u \circ v)$ . The special case of this with  $u = f$  gives us  $f(\Phi_{x_0}(v)) = \Phi_{x_0}(f \circ v)$  for all  $v \in M_f$ .

**Lemma 7.7** The mapping  $\Phi_{x_0}$  is a bijection.

*Proof* If  $g \in X$  then by Lemma 3.5 there exists a finite set  $A$  with  $x = \omega(A)$  and it follows that  $\Phi_{x_0}(f_A) = f_A(x_0) = \omega(A) = x$ . Thus  $\Phi_{x_0}$  is surjective. Now let  $u_1, u_2 \in M_f$  with  $\Phi_{x_0}(u_1) = \Phi_{x_0}(u_2)$ . By the definition of  $M_f$  there exist finite sets  $A$  and  $B$  with  $u_1 = f_A$  and  $u_2 = f_B$ , and hence

$$\omega(A) = \Phi_{x_0}(f_A) = \Phi_{x_0}(u_1) = \Phi_{x_0}(u_2) = \Phi_{x_0}(f_B) = \omega(B) .$$

Therefore by Proposition 7.1  $f_A = f_B$ , i.e.,  $u_1 = u_2$ , which shows that  $\Phi_{x_0}$  is also injective.  $\square$

*Proof of Theorem 6.1:* Since  $\Phi_{x_0} : M_f \rightarrow X$  is a bijection there clearly exists a unique binary relation  $\oplus$  on  $X$  such that

$$\Phi_{x_0}(u_1) \oplus \Phi_{x_0}(u_2) = \Phi_{x_0}(u_1 \circ u_2)$$

for all  $u_1, u_2 \in M_f$ . The operation  $\oplus$  is associative since  $\circ$  has this property: If  $x_1, x_2, x_3 \in X$  and  $u_1, u_2, u_3 \in M_f$  are such that  $x_j = \Phi_{x_0}(u_j)$  for each  $j$  then

$$\begin{aligned} (x_1 \oplus x_2) \oplus x_3 &= (\Phi_{x_0}(u_1) \oplus \Phi_{x_0}(u_2)) \oplus \Phi_{x_0}(u_3) \\ &= \Phi_{x_0}(u_1 \circ u_2) \oplus \Phi_{x_0}(u_3) = \Phi_{x_0}((u_1 \circ u_2) \circ u_3) \\ &= \Phi_{x_0}(u_1 \circ (u_2 \circ u_3)) = \Phi_{x_0}(u_1) \oplus \Phi_{x_0}(u_2 \circ u_3) \\ &= \Phi_{x_0}(u_1) \oplus (\Phi_{x_0}(u_2) \oplus \Phi_{x_0}(u_3)) = x_1 \oplus (x_2 \oplus x_3). \end{aligned}$$

In the same way  $\oplus$  is commutative, since by Lemma 7.6 the restriction of  $\circ$  to  $M_f$  has this property: If  $x_1, x_2 \in X$  and  $u_1, u_2 \in M_f$  are such that  $x_1 = \Phi_{x_0}(u_1)$  and  $x_2 = \Phi_{x_0}(u_2)$  then  $u_1 \circ u_2 = u_2 \circ u_1$  and so

$$\begin{aligned} x_1 \oplus x_2 &= \Phi_{x_0}(u_1) \oplus \Phi_{x_0}(u_2) \\ &= \Phi_{x_0}(u_1 \circ u_2) = \Phi_{x_0}(u_2 \circ u_1) = \Phi_{x_0}(u_2) \oplus \Phi_{x_0}(u_1) = x_2 \oplus x_1. \end{aligned}$$

Moreover, if  $x \in X$  and  $u \in M_f$  is such that  $x = \Phi_{x_0}(u)$  then

$$x \oplus x_0 = \Phi_{x_0}(u) \oplus \Phi_{x_0}(\text{id}_X) = \Phi_{x_0}(u \circ \text{id}_X) = \Phi_{x_0}(u) = x,$$

and so  $x \oplus x_0 = x$  for all  $x \in X$ .

Let  $x_1, x_2 \in X$ ; we next show that for some  $x \in X$  either  $x_1 = x_2 \oplus x$  or  $x_2 = x_1 \oplus x$ . Let  $u_1, u_2 \in M_f$  be such that  $x_1 = \Phi_{x_0}(u_1)$  and  $x_2 = \Phi_{x_0}(u_2)$  and let  $A$  and  $B$  be finite sets with  $u_1 = f_A$  and  $u_2 = f_B$ . By Theorem 2.4 there either exists an injective mapping  $p : A \rightarrow B$  or an injective mapping  $q : B \rightarrow A$ . Assume the former holds and put  $B' = p(A)$  and  $C = B \setminus B'$ . Then  $B'$  and  $C$  are disjoint and  $B = B' \cup C$ ; moreover,  $A \approx B'$  (since  $p$  considered as a mapping from  $A$  to  $B'$  is a bijection) and so by Proposition 7.2  $f_A = f_{B'}$ . Thus, putting  $x = \Phi_{x_0}(f_C)$ , it follows that

$$\begin{aligned} x_2 &= \Phi_{x_0}(u_2) = \Phi_{x_0}(f_B) = \Phi_{x_0}(f_{B' \cup C}) = \Phi_{x_0}(f_{B'} \circ f_C) \\ &= \Phi_{x_0}(f_{B'}) \oplus \Phi_{x_0}(f_C) = \Phi_{x_0}(f_A) \oplus \Phi_{x_0}(f_C) = \Phi_{x_0}(u_1) \oplus x = x_1 \oplus x. \end{aligned}$$

On the other hand, if there exists an injective mapping  $q : B \rightarrow A$  then the same argument shows there exists  $x \in X$  with  $x_1 = x_2 \oplus x$ .

Now to (a0) and (a1), and we have seen above that (a0) holds. Let  $x, x' \in X$  and let  $u, u' \in M_f$  with  $x = \Phi_{x_0}(u)$  and  $x' = \Phi_{x_0}(u')$ . Then by (#)

$$\begin{aligned} x \oplus f(x') &= \Phi_{x_0}(u) \oplus f(\Phi_{x_0}(u')) = \Phi_{x_0}(u) \oplus \Phi_{x_0}(f \circ u') \\ &= \Phi_{x_0}(u \circ f \circ u') = \Phi_{x_0}(f \circ u \circ u') = f(\Phi_{x_0}(u \circ u')) \\ &= f(\Phi_{x_0}(u) \oplus \Phi_{x_0}(u')) = f(x \oplus x') \end{aligned}$$

and so (a1) holds. If  $\oplus'$  is another binary operation on  $X$  satisfying (a0) and (a1) then it is easy to see that  $X_0 = \{x' \in X : x \oplus' x' = x \oplus x' \text{ for all } x \in X\}$  is a  $f$ -invariant subclass of  $X$  containing  $x_0$ . Hence  $X_0 = X$ , since  $\mathbb{I}$  is minimal, which implies that  $\oplus' = \oplus$ .

Finally, if  $A$  and  $B$  are disjoint finite sets then

$$\omega(A) \oplus \omega(B) = \Phi_{x_0}(f_A) \oplus \Phi_{x_0}(f_B) = \Phi_{x_0}(f_A \circ f_B) = \Phi_{x_0}(f_{A \cup B}) = \omega(A \cup B).$$

Moreover,  $\oplus$  is uniquely determined by this requirement: Consider any binary operation  $\oplus'$  on  $X$  for which  $\omega(A) \oplus' \omega(B) = \omega(A \cup B)$  whenever  $A$  and  $B$  are disjoint finite sets. If  $C$  and  $D$  are any finite sets then there exists a disjoint pair  $(A, B)$  with  $(A, B) \approx (C, D)$  and hence by Theorem 3.1

$$\omega(C) \oplus' \omega(D) = \omega(A) \oplus' \omega(B) = \omega(A \cup B) = \omega(A) \oplus \omega(B) = \omega(C) \oplus \omega(D)$$

and so by Lemma 3.5  $\oplus' = \oplus$ . This completes the proof of Theorem 6.1.  $\square$

Let  $\oplus$  be the operation given in Theorem 6.1. The theorem shows in particular that  $(X, \oplus, x_0)$  is a commutative monoid. The next result generalises Propositions 6.2 and 6.3 and shows how properties of the mapping  $f$  correspond to properties of the monoid  $(X, \oplus, x_0)$ .

**Proposition 7.3** (1)  $(X, \oplus, x_0)$  is a group if and only if  $f$  is a bijection.

(2) The cancellation law holds in  $(X, \oplus, x_0)$  if and only if  $f$  is injective.

*Proof* We have the commutative monoid  $(X, \oplus, x_0)$ , and also the commutative monoid  $(M_f, \circ, \text{id}_X)$ . Now the operation  $\oplus$  was defined so that

$$\Phi_{x_0}(u_1) \oplus \Phi_{x_0}(u_2) = \Phi_{x_0}(u_1 \circ u_2)$$

for all  $u_1, u_2 \in M_f$  and, since  $\Phi_{x_0}(\text{id}_X) = x_0$  and  $\Phi_{x_0}$  is a bijection, this means  $\oplus$  was defined to make  $\Phi_{x_0} : (M_f, \circ, \text{id}_X) \rightarrow (X, \oplus, x_0)$  a monoid isomorphism. It follows that the cancellation law holds in  $(X, \oplus, x_0)$  if and only if it holds in  $(M_f, \circ, \text{id}_X)$  and that  $(X, \oplus, x_0)$  will be a group if and only if  $(M_f, \circ, \text{id}_X)$  is. It is

thus enough to prove the statements in the proposition with  $(X, \oplus, x_0)$  replaced by  $(M_f, \circ, x_0)$ .

(1) We first show that  $u^{-1} \in M_f$  whenever  $u \in M_f$  is a bijection. This follows from the fact that  $u^{-1}(x_0) \in X$  and  $\Phi_{x_0}$  is surjective and so there exists  $v \in M_f$  with  $\Phi_{x_0}(v) = u^{-1}(x_0)$ ; thus by (#)

$$\Phi_{x_0}(u \circ v) = u(\Phi_{x_0}(v)) = u(u^{-1}(x_0)) = x_0 = \Phi_{x_0}(\text{id}_X)$$

and therefore  $u \circ v = \text{id}_X$ , since  $\Phi_{x_0}$  is injective. Hence  $u^{-1} = v \in M_f$ . Now clearly  $M_f$  is a group if and only if each mapping  $u \in M_f$  is a bijection and  $u^{-1} \in M_f$ , and we have just seen that  $u^{-1} \in M_f$  holds automatically whenever  $u \in M_f$  is a bijection. Moreover, by Lemma 7.5 (1) each element of  $M_f$  is a bijection if and only if  $f$  is a bijection.

(2) Suppose the cancellation law holds in  $(M_f, \circ, \text{id}_X)$ , and let  $x_1, x_2 \in X$  with  $f(x_1) = f(x_2)$ . Then there exist  $u_1, u_2 \in M_f$  with  $\Phi_{x_0}(u_1) = x_1$  and  $\Phi_{x_0}(u_2) = x_2$  (since  $\Phi_{x_0}$  is surjective), and hence by (#)

$$\Phi_{x_0}(f \circ u_1) = f(\Phi_{x_0}(u_1)) = f(x_1) = f(x_2) = f(\Phi_{x_0}(u_2)) = \Phi_{x_0}(f \circ u_2).$$

It follows that  $f \circ u_1 = f \circ u_2$  (since  $\Phi_{x_0}$  is injective) and so  $u_1 = u_2$ . In particular  $x_1 = x_2$ , which shows that  $f$  is injective. The converse is immediate, since if  $f$  is injective then by Lemma 7.5 (2) so is each  $u \in M_f$  and hence  $u_1 = u_2$  whenever  $u \circ u_1 = u \circ u_2$ .  $\square$

We now begin the preparations for the proof of Theorem 6.2.

For each  $u \in M_f$  consider the iterator  $\mathbb{I}_*^u = (\mathbb{T}_X, u_*, \text{id}_X)$ , where  $u_* : \mathbb{T}_X \rightarrow \mathbb{T}_X$  is defined by  $u_*(v) = u \circ v$  for all  $v \in \mathbb{T}_X$  and let  $\omega_*^u$  be the assignment of finite sets in  $\mathbb{I}_*^u$ . Thus  $\omega_*^u : \mathbf{Fin} \rightarrow \mathbb{T}_X$  is the unique mapping with  $\omega_*^u(\emptyset) = \text{id}_X$  such that  $\omega_*^u(A \cup \{a\}) = u_*(\omega_*^u(A)) = u \circ \omega_*^u(A)$  for each finite set  $A$  and each  $a \notin A$ .

Now it is more convenient to write  $u_A$  instead of  $\omega_*^u(A)$  (this being consistent with the previous notation for the special case with  $u = f$ ). Thus  $A \mapsto u_A$  is the unique mapping with  $u_\emptyset = \text{id}_X$  such that  $u_{(A \cup \{a\})} = u \circ u_A$  for each finite set  $A$  and each  $a \notin A$ .

**Lemma 7.8** (1)  $(f_B)_A = f_{B \times A}$  for all finite sets  $A$  and  $B$ .

(2)  $(f_B)_A = (f_A)_B$  for all finite sets  $A$  and  $B$ .

*Proof* (1) Let  $A$  and  $B$  be finite sets and put  $\mathcal{S} = \{C \in \mathcal{P}(A) : (f_B)_C = f_{B \times C}\}$ . Then  $\emptyset \in \mathcal{S}$  since  $(f_B)_\emptyset = \text{id}_X = f_\emptyset = f_{B \times \emptyset}$ . Let  $C \in \mathcal{S}$  (and so  $(f_B)_C = f_{B \times C}$ )

and let  $a \in A \setminus C$ . Then  $B \times (C \cup \{a\})$  is the disjoint union of the sets  $B \times \{a\}$  and  $B \times C$  and  $B \times \{a\} \approx B$ ; thus by Proposition 7.2 and Lemma 7.4

$$(f_B)_{C \cup \{a\}} = f_B \circ (f_B)_C = f_B \circ f_{B \times C} = f_{B \times \{a\}} \circ f_{B \times C} = f_{(B \times \{a\}) \cup (B \times C)} = f_{B \times (C \cup \{a\})}$$

and so  $C \cup \{a\} \in \mathcal{S}$ . Hence  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Hence  $(f_B)_A = f_{B \times A}$ .

(2) By Proposition 7.2  $f_{B \times A} = f_{A \times B}$ , since clearly  $B \times A \approx A \times B$ , and therefore by (1)  $(f_B)_A = f_{B \times A} = f_{A \times B} = (f_A)_B$ .  $\square$

The next result is not needed in what follows, but it shows that ( $\spadesuit$ ) in Section 6 holds, and could thus be used instead of Lemma 6.3 in the previous proof of Theorem 6.2.

**Lemma 7.9** Let  $(A, B)$  and  $(A', B')$  be pairs of finite sets.

- (1) If  $f_A = f_{A'}$  and  $f_B = f_{B'}$  then  $f_{A \times B} = f_{A' \times B'}$ .
- (2) If  $\omega(A, B) = \omega(A', B')$  then  $\omega(A \times B) = \omega(A' \times B')$ .

*Proof* (1) By several applications of Lemma 7.8 (1) and (2) we have

$$f_{A \times B} = (f_B)_A = (f_{B'})_A = (f_A)_{B'} = (f_{A'})_{B'} = (f_{B'})_{A'} = f_{A' \times B'}.$$

(2) This follows immediately from (1) and Proposition 7.1.  $\square$

**Lemma 7.10** Let  $v \in M_f$ . Then:

- (1)  $v_A \in M_f$  for every finite set  $A$ .
- (2) If  $A$  and  $B$  are finite sets with  $f_A = f_B$  then  $v_A = v_B$ .

*Proof* (1) Let  $A$  be a finite set and put  $\mathcal{S} = \{B \in \mathcal{P}(A) : v_B \in M_f\}$ . Then  $\emptyset \in \mathcal{S}$ , since  $v_\emptyset = \text{id}_X \in M_f$ . Consider  $B \in \mathcal{S}^p$  (and so  $v_B \in M_f$ ) and let  $a \in A \setminus B$ . Then  $v_{B \cup \{a\}} = v \circ v_B \in M_f$ , since  $M_f$  is a submonoid of  $(T_X, \circ, \text{id}_X)$ , and so  $B \cup \{a\} \in \mathcal{S}$ . Hence  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . Thus  $v_A \in M_f$ .

(2) There exists a finite set  $C$  with  $v = f_C$  and thus by Lemma 7.8 (2)

$$v_A = (f_C)_A = (f_A)_C = (f_B)_C = (f_C)_B = v_B. \quad \square$$

Let  $v \in M_f$ ; then by Lemma 7.10 there exists a unique mapping  $\psi_v : M_f \rightarrow M_f$  such that  $\psi_v(f_A) = v_A$  for each finite set  $A$ , and in particular  $\psi_v(f) = v$  (since if  $a$  is any element then  $f = f_{\{a\}}$  and  $v = v_{\{a\}}$ ). Moreover, if  $v = f_B$  then by Lemma 7.8 (1)  $\psi_v(f_A) = f_{A \times B}$ .

A mapping  $\psi : M_f \rightarrow M_f$  is an endomorphism (of the monoid  $(M_f, \circ, \text{id}_X)$ ) if  $\psi(\text{id}_X) = \text{id}_X$  and  $\psi(u_1 \circ u_2) = \psi(u_1) \circ \psi(u_2)$  for all  $u_1, u_2 \in M_f$ .

**Lemma 7.11** (1)  $\psi_v$  is an endomorphism for each  $v \in M_f$ .

(2)  $\psi_v(u) = \psi_u(v)$  for all  $u, v \in M_f$ .

*Proof* (1) If  $u_1, u_2 \in M_f$  then there exist disjoint finite sets  $A$  and  $B$  with  $u_1 = f_A$  and  $u_2 = f_B$  and hence by Lemma 7.4

$$\psi_v(u_1 \circ u_2) = \psi_v(f_A \circ f_B) = \psi_v(f_{A \cup B}) = v_{A \cup B} = v_A \circ v_B = \psi_v(f_A) \circ \psi_v(f_B)$$

(noting that proof of Lemma 7.4 also shows that  $v_{A \cup B} = v_A \circ v_B$ . Moreover, we have  $\psi_v(\text{id}_X) = \psi_v(f_\emptyset) = v_\emptyset = \text{id}_X$  and hence  $\psi_v$  is an endomorphism.

(2) Let  $A, B$  be finite sets with  $u = f_A$  and  $v = f_B$ . Then by Lemma 7.8 (2)

$$\psi_v(u) = \psi_v(f_A) = v_A = (f_B)_A = (f_A)_B = u_B = \psi_u(f_B) = \psi_u(v). \quad \square$$

*Proof of Theorem 6.2:* Define a binary operation  $\diamond : M_f \times M_f \rightarrow M_f$  by letting

$$u \diamond v = \psi_u(v)$$

for all  $u, v \in M_f$ . In particular, if  $A$  and  $B$  are finite sets then by Lemma 7.8 (1)  $f_A \diamond f_B = \psi_{f_A}(f_B) = (f_A)_B = f_{A \times B}$  and therefore

$$f_A \diamond f_B = f_{A \times B}$$

for all finite sets  $A$  and  $B$ . Let  $u, v, w \in M_f$  and  $A, B$  and  $C$  be finite sets with  $u = f_A, v = f_B$  and  $w = f_C$ . Then clearly  $A \times (B \times C) \approx (A \times B) \times C$ , so by Proposition 7.2  $f_{A \times (B \times C)} = f_{(A \times B) \times C}$  and thus

$$\begin{aligned} u \diamond (v \diamond w) &= f_A \diamond (f_B \diamond f_C) = f_A \diamond f_{B \times C} \\ &= f_{A \times (B \times C)} = f_{(A \times B) \times C} = f_{A \times B} \diamond f_C = (f_A \diamond f_B) \diamond f_C = (u \diamond v) \diamond w. \end{aligned}$$

Hence  $\diamond$  is associative. Moreover, Lemma 7.11 (2) shows that  $\diamond$  is commutative, since  $u \diamond v = \psi_u(v) = \psi_v(u) = v \diamond u$  for all  $u, v \in M_f$ . Also (with a any element)  $u \diamond f = u \diamond f_{\{a\}} = u_{\{a\}} = u$ , i.e.,  $u \diamond f = u$  for all  $u \in M_f$ , and by Lemma 7.11 (1)  $u \diamond \text{id}_X = \text{id}_X$  and  $u \diamond (v_1 \circ v_2) = (u \diamond v_1) \circ (u \diamond v_2)$  for all  $u, v_1, v_2 \in M_f$ .

Now since  $\Phi_{x_0} : M_f \rightarrow X$  is a bijection there clearly exists a unique binary relation  $\otimes$  on  $X$  such that

$$\Phi_{x_0}(u_1) \otimes \Phi_{x_0}(u_2) = \Phi_{x_0}(u_1 \diamond u_2)$$

for all  $u_1, u_2 \in M_f$ , and exactly as in the proof of Theorem 6.1 the operation  $\otimes$  is associative and commutative since  $\diamond$  has these properties. The same holds

true of the distributive law: Let  $x, x_1, x_2 \in X$ , and  $u, v_1, v_2 \in M_f$  be such that  $x = \Phi_{x_0}(u)$ ,  $x_1 = \Phi_{x_0}(v_1)$  and  $x_2 = \Phi_{x_0}(v_2)$ . Then

$$\begin{aligned} x \otimes (x_1 \oplus x_2) &= \Phi_{x_0}(u) \otimes (\Phi_{x_0}(v_1) \oplus \Phi_{x_0}(v_2)) \\ &= \Phi_{x_0}(u) \otimes \Phi_{x_0}(v_1 \circ v_2) = \Phi_{x_0}(u \diamond (v_1 \circ v_2)) \\ &= \Phi_{x_0}((u \diamond v_1) \circ (u \diamond v_2)) = \Phi_{x_0}(u \diamond v_1) \oplus \Phi_{x_0}(u \diamond v_2) \\ &= (\Phi_{x_0}(u) \otimes \Phi_{x_0}(v_1)) \oplus (\Phi_{x_0}(u) \otimes \Phi_{x_0}(v_2)) = (x \otimes x_1) \oplus (x \otimes x_2) \end{aligned}$$

Next, if  $x \in X$  and  $u \in M_f$  is such that  $x = \Phi_{x_0}(u)$  then

$$\begin{aligned} x \otimes x_0 &= \Phi_{x_0}(u) \otimes \Phi_{x_0}(\text{id}_X) = \Phi_{x_0}(u \diamond \text{id}_X) = \Phi_{x_0}(\text{id}_X) = x_0, \\ x \otimes f(x_0) &= \Phi_{x_0}(u) \otimes \Phi_{x_0}(f) = \Phi_{x_0}(u \diamond f) = \Phi_{x_0}(u) = x \end{aligned}$$

and so  $x \otimes x_0 = x_0$  and  $x \otimes f(x_0) = x$  for all  $x \in X$ .

We have already seen (m0) holds and, since  $f(x_0)$  is an identity element, (m1) is a special case of the distributive law: Let  $x, x' \in X$ ; then by (a0) and (a1) and since  $\oplus$  is commutative it follows that  $f(x') = f(x' \oplus x_0) = x' \oplus f(x_0) = f(x_0) \oplus x'$ , and hence  $x \otimes f(x') = x \otimes (f(x_0) \oplus x') = (x \otimes f(x_0)) \oplus (x \otimes x') = x \oplus (x \otimes x')$ , which is (m1). Finally, if  $\otimes'$  is another binary operation satisfying (m0) and (m1) then it is easy to see that  $X_0 = \{x' \in X : x \otimes' x' = x \otimes x' \text{ for all } x \in X\}$  is a  $f$ -invariant subset of  $X$  containing  $x_0$ . Hence  $X_0 = X$ , since  $\mathbb{I}$  is minimal, which implies that  $\otimes' = \otimes$ .  $\square$



## 8 Permutations

Let  $E$  be a set; as in Section 7  $T_E$  denotes the set of all mappings  $f : E \rightarrow E$  of  $E$  into itself, considered as a monoid with functional composition  $\circ$  as monoid operation and  $\text{id}_E$  as identity element. Denote by  $S_E$  the set of bijective mappings in  $T_E$ , thus  $S_E$  is a submonoid of  $T_E$  and it is a group.

If  $A$  is finite then the elements of  $S_A$  are often referred to as permutations.

An element  $\tau$  of  $S_E$  will be called an  $E$ -transposition, or just a transposition when it is clear which set  $E$  is involved, if there exist  $b, c \in E$  with  $b \neq c$  such that

$$\tau(x) = \begin{cases} c & \text{if } x = b, \\ b & \text{if } x = c, \\ x & \text{otherwise.} \end{cases}$$

This transposition will be denoted by  $\tau_{b,c}$ , or by  $\tau_{b,c}^E$  when the set  $E$  cannot be determined from the context. Each transposition is its own inverse.

Denote by  $F_2$  the multiplicative group  $\{+, -\}$  with  $+\cdot+ = -\cdot- = +$  and  $-\cdot+ = +\cdot- = -$ . For each element  $s \in F_2$  the other element will be denoted by  $-s$ .

Let  $E$  be a set; we call a mapping  $\sigma_E : S_E \rightarrow F_2$  an  $E$ -signature if  $\sigma_E(\text{id}_E) = +$  and  $\sigma_E(\tau \circ f) = -\sigma_E(f)$  for each  $f \in S_E$  and for each  $E$ -transposition  $\tau$ . In particular, it then follows that  $\sigma_E(\tau) = -$  for each  $E$ -transposition  $\tau$ .

**Theorem 8.1** For each finite set  $A$  there exists a unique  $A$ -signature  $\sigma : S_A \rightarrow F_2$ . Moreover,  $\sigma(f \circ g) = \sigma(f) \cdot \sigma(g)$  for all  $f, g \in S_A$  and hence  $\sigma$  is a group homomorphism.

*Proof* We first need some preparation and start by noting some of the standard identities concerning the composition of transpositions which will be needed.

**Lemma 8.1** Let  $p, q, r, s$  be elements of some set  $E$  with  $p \neq q$  and  $r \neq s$ . Then

- (1)  $\tau_{r,s} \circ \tau_{p,q} = \tau_{p,q} \circ \tau_{r,s}$  if the elements  $p, q, r, s$  are all different.
- (2)  $\tau_{r,s} \circ \tau_{p,q} = \tau_{q,s} \circ \tau_{r,s}$  if  $p = r$  and  $q \neq s$ .
- (3)  $\tau_{r,s} \circ \tau_{p,q} = \text{id}_E = \tau_{p,q} \circ \tau_{r,s}$  if  $p = r$  and  $q = s$ .

*Proof* Just check what happens to the elements  $p, q, r, s$ . (All other elements in  $E$  remain fixed.)  $\square$

In what follows let  $B$  be a set and  $a \notin B$ ; put  $C = B \cup \{a\}$ . Denote by  $S_C^a$  the subgroup of  $S_C$  consisting of those mappings  $f \in S_C$  with  $f(a) = a$ . For each  $f \in S_C^a$  denote by  $f_B$  the restriction of  $f$  to  $B$ , considered as an element of  $S_B$ . The mapping  $\varphi_B : S_C^a \rightarrow S_B$  with  $\varphi_B(f) = f_B$  for each  $f \in S_C^a$  is clearly a group isomorphism.

Going in the other direction, if  $g \in S_B$  then denote by  $g^a$  the extension of  $g$  to  $C$  with  $g^a(a) = a$ ; thus  $g^a \in S_C^a$  and  $(g^a)_B = g$ . The mapping  $\psi^a : S_B \rightarrow S_C^a$  with  $\psi^a(g) = g^a$  for each  $g \in S_B$  is the inverse of the isomorphism  $\varphi_B$ .

Note that if  $b, b' \in B$  with  $b \neq b'$  then  $\tau_{b,b'}^C \in S_C^a$  and  $\varphi_B(\tau_{b,b'}^C) = \tau_{b,b'}^B$ .

Consider  $f \in S_C \setminus S_C^a$ ; then  $b = f(a) \in B$  and  $\tau_{a,b}^C \circ f \in S_C^a$ . Since  $\tau_{a,b}^C$  is its own inverse we have  $f = \tau_{a,b}^C \circ (\tau_{a,b}^C \circ f)$  which shows that each mapping in  $S_C \setminus S_C^a$  can be written as the composition of a mapping in  $S_C^a$  with a  $C$ -transposition. The transposition can also be chosen to be on the other side of a mapping from  $S_C^a$ : There exists a unique element  $b' \in B$  with  $f(b') = a$ , then  $f \circ \tau_{a,b'}^C \in S_C^a$  and  $f = (f \circ \tau_{a,b'}^C) \circ \tau_{a,b'}^C$ .

**Lemma 8.2** Let  $\sigma_C : S_C \rightarrow F_2$  be a  $C$ -signature and define  $\sigma_B : S_B \rightarrow F_2$  by  $\sigma_B = \sigma_C \circ \psi^a$ . Then  $\sigma_B$  is a  $B$ -signature. Moreover,

$$\begin{aligned} \sigma_C(f) &= \sigma_B(\varphi_B(f)) \text{ for each } f \in S_C^a, \\ \sigma_C(f) &= -\sigma_B(\varphi_B(\tau_{a,b}^C \circ f)) \text{ for each } f \in S_C \setminus S_C^a, \text{ where } b = f(a) \in B \text{ (and as} \\ &\quad \text{above } \tau_{a,b}^C \circ f \in S_C^a). \end{aligned}$$

In particular,  $\sigma_C$  is uniquely determined by  $\sigma_B$ .

*Proof* We have  $\sigma_B(\text{id}_B) = \sigma_C(\psi^a(\text{id}_B)) = \sigma_C(\text{id}_C) = +$  and

$$\sigma_B(\tau \circ f) = \sigma_C(\psi^a(\tau \circ f)) = \sigma_C(\psi^a(\tau) \circ \psi^a(f)) = -\sigma_C(\psi^a(f)) = -\sigma_B(f)$$

for each  $f \in S_B$  and for each  $B$ -transposition  $\tau$ , since  $\psi^a(\tau)$  is a  $C$ -transposition. This shows that  $\sigma_B$  is a  $B$ -signature. If  $f \in S_C^a$  then  $f = \psi^a(\varphi_B(f))$  and hence  $\sigma_C(f) = \sigma_C(\psi^a(\varphi_B(f))) = \sigma_B(\varphi_B(f))$ . If  $f \in S_C \setminus S_C^a$  then  $f = \tau_{a,b}^C \circ (\tau_{a,b}^C \circ f)$  and  $\tau_{a,b}^C \circ f \in S_C^a$ . Thus  $\sigma_C(f) = -\sigma_C(\tau_{a,b}^C \circ f) = -\sigma_B(\varphi_B(\tau_{a,b}^C \circ f))$ .  $\square$

**Lemma 8.3** Let  $\sigma_B : S_B \rightarrow F_2$  be a  $B$ -signature and let  $\sigma_C : S_C \rightarrow F_2$  be the mapping given by

$$\begin{aligned} \sigma_C(f) &= \sigma_B(\varphi_B(f)) \text{ for each } f \in S_C^a, \\ \sigma_C(f) &= -\sigma_B(\varphi_B(\tau_{a,b}^C \circ f)) \text{ for each } f \in S_C \setminus S_C^a, \text{ where } b = f(a). \end{aligned}$$

Then  $\sigma_C$  is a  $C$ -signature.

*Proof* To start with  $\sigma_C(\text{id}_C) = \sigma_B(\varphi_B(\text{id}_C)) = \sigma_B(\text{id}_B) = +$ . Thus let  $f \in S_C$  and let  $c, d \in C$  with  $c \neq d$ ; we must show that  $\sigma_C(\tau_{c,d}^C \circ f) = -\sigma_C(f)$ .

Suppose first that both  $c$  and  $d$  lie in  $B$ . There are three cases:

(1)  $f \in S_C^a$ . In this case  $\tau_{c,d}^C \circ f \in S_C^a$  and hence

$$\begin{aligned} \sigma_C(\tau_{c,d}^C \circ f) &= \sigma_B(\varphi_B(\tau_{c,d}^C \circ f)) = \sigma_B(\varphi_B(\tau_{c,d}^C) \circ \varphi_B(f)) \\ &= \sigma_B(\tau_{c,d}^B \circ \varphi_B(f)) = -\sigma_B(\varphi_B(f)) = -\sigma_C(f). \end{aligned}$$

(2)  $f \in S_C \setminus S_C^a$  with  $b = f(a) \notin \{c, d\}$ . In this case we have  $\tau_{c,d}^C \circ f \in S_C \setminus S_C^a$  with  $(\tau_{c,d}^C \circ f)(a) = b$  and by Lemma 8.1  $\tau_{a,b}^C \circ \tau_{c,d}^C = \tau_{c,d}^C \circ \tau_{a,b}^C$ , since the elements  $a, b, c, d$  are all different. Hence

$$\begin{aligned} \sigma_C(\tau_{c,d}^C \circ f) &= -\sigma_B(\varphi_B(\tau_{a,b}^C \circ \tau_{c,d}^C \circ f)) = -\sigma_B(\varphi_B(\tau_{c,d}^C \circ \tau_{a,b}^C \circ f)) \\ &= -\sigma_B(\varphi_B(\tau_{c,d}^C) \circ \varphi_B(\tau_{a,b}^C \circ f)) = -\sigma_B(\tau_{c,d}^B \circ \varphi_B(\tau_{a,b}^C \circ f)) \\ &= \sigma_B(\varphi_B(\tau_{a,b}^C \circ f)) = -\sigma_C(f). \end{aligned}$$

(3)  $f \in S_C \setminus S_C^a$  with  $b = f(a) \in \{c, d\}$ , and without loss of generality assume  $b = d$ . In this case  $\tau_{c,d}^C \circ f \in S_C \setminus S_C^a$  with  $(\tau_{c,d}^C \circ f)(a) = c$  and by Lemma 8.1 (2)  $\tau_{c,d}^C \circ \tau_{a,b}^C = \tau_{a,c}^C \circ \tau_{c,d}^C$ . Hence

$$\begin{aligned} \sigma_C(\tau_{c,d}^C \circ f) &= -\sigma_B(\varphi_B(\tau_{a,c}^C \circ \tau_{c,d}^C \circ f)) = -\sigma_B(\varphi_B(\tau_{c,d}^C \circ \tau_{a,b}^C \circ f)) \\ &= -\sigma_B(\varphi_B(\tau_{c,d}^C) \circ \varphi_B(\tau_{a,b}^C \circ f)) = -\sigma_B(\tau_{c,d}^B \circ \varphi_B(\tau_{a,b}^C \circ f)) \\ &= \sigma_B(\varphi_B(\tau_{a,b}^C \circ f)) = -\sigma_C(f). \end{aligned}$$

This deals with the cases when both  $c$  and  $d$  lie in  $B$ . Suppose now then that one of  $c$  and  $d$  is equal to  $a$ , and without loss of generality it can be assumed that  $c = a$  (and so  $d \in B$ ). There are the same three cases as above:

(1)  $f \in S_C^a$ . Here  $\tau_{c,d}^C \circ f \in S_C \setminus S_C^a$  with  $(\tau_{c,d}^C \circ f)(a) = d$  and thus

$$\sigma_C(\tau_{c,d}^C \circ f) = -\sigma_B(\varphi_B(\tau_{a,d}^C \circ \tau_{c,d}^C \circ f)) = -\sigma_B(\varphi_B(f)) = -\sigma_C(f),$$

since  $\tau_{a,d}^C \circ \tau_{c,d}^C = \tau_{c,d}^C \circ \tau_{a,d}^C = \text{id}_C$ .

(2)  $f \in S_C \setminus S_C^a$  with  $b = f(a) \neq d$ . Here  $\tau_{c,d}^C \circ f \in S_C \setminus S_C^a$  with  $(\tau_{c,d}^C \circ f)(a) = b$  and by Lemma 8.1 (2)  $\tau_{a,b}^C \circ \tau_{c,d}^C = \tau_{b,d}^C \circ \tau_{a,b}^C$ . Hence

$$\begin{aligned} \sigma_C(\tau_{c,d}^C \circ f) &= -\sigma_B(\varphi_B(\tau_{a,b}^C \circ \tau_{c,d}^C \circ f)) = -\sigma_B(\varphi_B(\tau_{b,d}^C \circ \tau_{a,b}^C \circ f)) \\ &= -\sigma_B(\varphi_B(\tau_{b,d}^C) \circ \varphi_B(\tau_{a,b}^C \circ f)) = -\sigma_B(\tau_{b,d}^B \circ \varphi_B(\tau_{a,b}^C \circ f)) \\ &= \sigma_B(\varphi_B(\tau_{a,b}^C \circ f)) = -\sigma_C(f). \end{aligned}$$

(3)  $f \in S_C \setminus S_C^a$  with  $b = f(a) = d$ , and hence  $(a, b) = (c, d)$ . Here  $\tau_{c,d}^C \circ f \in S_C^a$ , since  $(\tau_{c,d}^C \circ f)(a) = a$  and therefore

$$\sigma_C(\tau_{c,d}^C \circ f) = \sigma_B(\varphi_B(\tau_{c,d}^C \circ f)) = \sigma_B(\varphi_B(\tau_{a,b}^C \circ f)) = -\sigma_C(f).$$

This deals with the cases where one of  $c$  and  $d$  is equal to  $a$ , and so all of the possibilities have now been exhausted. Thus  $\sigma_C$  is a  $C$ -signature.  $\square$

The first statement in Theorem 8.1 follows directly from Lemmas 8.2 and 8.3: Let  $A$  be a finite set and let  $\mathcal{S}$  be the set consisting of those  $B \in \mathcal{P}(A)$  for which there exists a unique  $B$ -signature. Then  $\emptyset \in \mathcal{S}$ , since  $S_\emptyset = \{\text{id}_\emptyset\}$  (and there are no  $\emptyset$ -transpositions). Now let  $B \in \mathcal{S}^p$ , let  $a \in A \setminus B$  and put  $C = B \cup \{a\}$ . Then by Lemma 8.3 there exists a  $C$ -signature  $\sigma_C$  which is the unique  $C$ -signature, since by Lemma 8.2 it is uniquely determined by the unique  $B$ -signature  $\sigma_B$ . Thus  $B \cup \{a\} \in \mathcal{S}$ . Hence  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . This shows that for each finite set  $A$  there is a unique  $A$ -signature  $\sigma : S_A \rightarrow F_2$ .

In order to show that the second statement in Theorem 8.1 holds (i.e., that the unique  $A$ -signature is group homomorphism) we need the following fact:

**Proposition 8.1** *For each finite set  $A$  the group  $S_A$  is the least submonoid of  $T_A$  containing the  $A$ -transpositions.*

*Proof* Let  $A$  be a finite set and let  $\mathcal{S}$  be the set consisting of those  $B \in \mathcal{P}(A)$  for which  $S_B$  is the least submonoid of  $T_B$  containing the  $B$ -transpositions. Then  $\emptyset \in \mathcal{S}$ , since  $S_\emptyset = T_\emptyset = \{\text{id}_\emptyset\}$  (and there are no  $\emptyset$ -transpositions).

Now let  $B \in \mathcal{S}^p$  and  $a \in A \setminus B$ ; put  $C = B \cup \{a\}$  and consider any submonoid  $M$  of  $T_C$  containing the  $C$ -transpositions. Then  $S_C^a \cap M$  is a submonoid of  $T_C$  containing all  $C$ -transpositions of the form  $\tau_{b,c}^C$  with  $b, c \in B$  and hence  $\varphi_B(S_C^a \cap M)$  is a submonoid of  $T_B$  containing all the  $B$ -transpositions. Thus  $S_B \subset \varphi_B(S_C^a \cap M)$  (since  $B \in \mathcal{S}$ ) and it follows that  $S_C^a \subset M$ . But we have seen that each element of  $S_C \setminus S_C^a$  can be written in the form  $\tau \circ f$  with  $\tau$  a  $C$ -transposition and  $f \in S_C^a$  and hence also  $S_C \setminus S_C^a \subset M$ . This shows  $S_C = (S_C \setminus S_C^a) \cup S_C^a \subset M$ , i.e., that  $B \cup \{a\} \in \mathcal{S}$ .

Hence  $\mathcal{S}$  is an inductive  $A$ -system and so  $A \in \mathcal{S}$ . For each finite set  $A$  the group  $S_B$  is thus the least submonoid of  $T_B$  containing the  $B$ -transpositions.  $\square$

We also need the following standard fact:

**Lemma 8.4** *Let  $(M, \bullet, e)$  be a monoid and let  $T \subset M$ . If  $Q$  is any subset of  $M$  containing  $e$  such that  $t \bullet q \in Q$  for all  $q \in Q$  and all  $t \in T$  then  $\langle T \rangle \subset Q$ , where  $\langle T \rangle$  denotes the least submonoid of  $M$  containing  $T$ .*

*Proof* Let  $N = \{a \in M : a \bullet q \in Q \text{ for all } q \in Q\}$ ; then clearly  $e \in N$  and if  $a_1, a_2 \in N$  then  $(a_1 \bullet a_2) \bullet q = a_1 \bullet (a_2 \bullet q) \in B$  for all  $q \in Q$ , i.e.,  $a_1 \bullet a_2 \in N$ . Thus  $N$  is a submonoid of  $M$  and by assumption  $T \subset N$ ; hence  $\langle T \rangle \subset N$ . But  $N \subset Q$ , since  $e \in Q$ , and therefore  $\langle T \rangle \subset B$ .  $\square$

Let  $A$  be a finite set and consider the unique  $A$ -signature  $\sigma_A : S_A \rightarrow F_2$ . Let  $T$  be the set of  $A$ -transpositions and

$$Q = \{f \in S_A : \sigma_A(f \circ g) = \sigma_A(f) \cdot \sigma_A(g) \text{ for all } g \in S_A\};$$

thus in particular  $\text{id}_A \in Q$ . If  $\tau \in T$  and  $f \in Q$  then

$$\sigma_A((\tau \circ f) \circ g) = \sigma_A(\tau \circ (f \circ g)) = -\sigma_A(f \circ g) = -\sigma_A(f) \cdot \sigma_A(g) = \sigma_A(\tau \circ f) \cdot \sigma_A(g)$$

for all  $g \in S_A$  and therefore  $\tau \circ f \in Q$ . Hence by Lemma 8.4  $\langle T \rangle \subset Q$ . But by Proposition 8.1  $\langle T \rangle = S_A$  and so  $Q = S_A$ . This shows that the unique  $A$ -signature  $\sigma$  is a group homomorphism, which completes the proof of Theorem 8.1.  $\square$

## 9 The generalised associative law

Let  $\bullet$  be a binary operation on a set  $X$ , written using infix notation, so  $x_1 \bullet x_2$  denotes the product of  $x_1$  and  $x_2$ . The large majority of such operations occurring in mathematics are associative, meaning that  $(x_1 \bullet x_2) \bullet x_3 = x_1 \bullet (x_2 \bullet x_3)$  for all  $x_1, x_2, x_3 \in X$ . If  $\bullet$  is associative and  $x_1, x_2, \dots, x_n \in X$  then the product  $x_1 \bullet x_2 \bullet \dots \bullet x_n$  is well-defined, meaning its value does not depend on the order in which the operations are carried out.

This result will be established in the present section. We first define a particular order of carrying out the operations. This is the order in which at each stage the product of the current first and second components are taken. For example, the product of the 6 components  $x_1, x_2, x_3, x_4, x_5, x_6$  evaluated using this order results in the value  $\bullet(x_1, \dots, x_6) = (((((x_1 \bullet x_2) \bullet x_3) \bullet x_4) \bullet x_5) \bullet x_6)$ . In general, the corresponding product of  $n$  terms will be denoted by  $\bullet(x_1, \dots, x_n)$ .

Theorem 9.1 states that if  $\bullet$  is associative then

$$\bullet(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = \alpha \bullet \beta ,$$

where  $\alpha = \bullet(x_1, \dots, x_m)$  and  $\beta = \bullet(x_{m+1}, \dots, x_n)$ . This is a weak form of the generalised associative law, although it is one which is often all that is needed.

Theorem 9.2 gives the general form of the generalised associative law and states that if  $\bullet$  is associative then  $\bullet(x_1, \dots, x_n) = \bullet_{\mathbb{R}}(x_1, \dots, x_n)$  for each  $\mathbb{R}$  from the set of prescriptions describing how the operations are carried out. The main task is to give a rigorous definition of this set. We do this using partitions of intervals of the form  $\{k \in \mathbb{Z} : m \leq k \leq n\}$  in which each element in the partition is also an interval of this form.

Recall that by a partition of a set  $S$  we mean a subset  $\mathcal{Q}$  of  $\mathcal{P}_0(S)$  such that for each  $s \in S$  there exists a unique  $Q \in \mathcal{Q}$  such that  $s \in Q$ . Thus, different elements in a partition of  $S$  are disjoint and their union is  $S$ .

Consider the product  $((x_1 \bullet x_2) \bullet ((x_3 \bullet x_4) \bullet x_5))$ . The order of operations involved here can be described with the help of the following sequence of partitions of the set  $\{1, 2, 3, 4, 5\}$ :

$$\begin{aligned} &\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\} \\ &\{\{1\}, \{2\}, \{3, 4\}, \{5\}\} \\ &\{\{1, 2\}, \{3, 4\}, \{5\}\} \\ &\{\{1, 2\}, \{3, 4, 5\}\} \\ &\{\{1, 2, 3, 4, 5\}\} \end{aligned}$$

For each of these partitions (except the last one) the next partition is obtained by amalgamating two adjacent partitions. Corresponding to these partitions there is a sequence of partial evaluations:

$$\begin{aligned}
& \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\} \\
& \{\{x_1\}, \{x_2\}, \{(x_3 \bullet x_4)\}, \{x_5\}\} \\
& \{\{(x_1 \bullet x_2)\}, \{(x_3 \bullet x_4)\}, \{x_5\}\} \\
& \{\{(x_1 \bullet x_2)\}, \{((x_3 \bullet x_4) \bullet x_5)\}\} \\
& \{\{((x_1 \bullet x_2) \bullet ((x_3 \bullet x_4) \bullet x_5))\}\}
\end{aligned}$$

and the final expression is essentially the product we started with.

Instead of using intervals of the form  $\{k \in \mathbb{Z} : m \leq k \leq n\}$  we prefer to use a formulation in terms of a finite totally ordered set.

Thus let  $(E, \leq_E)$  be a non-empty finite total ordered set with maximum element  $z$  and define a mapping  $f : E \rightarrow E$  by letting  $f(e)$  be the least element in  $\{e' \in E : e' >_E e\}$  if  $e \neq z$  and putting  $f(z) = z$ . Then  $\mathbb{I} = (E, f, e_0)$ , with  $e_0$  the minimum element in  $E$ , is a finite iterator with fixed point  $z$  and by Proposition 5.2  $\mathbb{I}$  is minimal. The totally ordered set  $(E, \leq)$  is considered fixed in what follows, but recall that if  $A$  is any finite non-empty set then by Lemma 2.6 there exists a totally ordered set  $(E, \leq)$  with  $E \approx A$ .

For all  $r, s \in E$  with  $r \leq_E s$  put  $[r, s] = \{e \in E : r \leq_E e \leq_E s\}$ . Sets of this form will be called intervals. Also put  $[r, s) = \{e \in E : r \leq_E e <_E s\}$ .

**Lemma 9.1** *Let  $[p, q]$  be an interval and let  $B \subset [p, q]$ . Suppose  $p \in B$  and  $f(s) \in B$  for all  $s \in [p, q]$ . Then  $B = [p, q]$ .*

*Proof* Consider the restriction  $\leq_{[p, q]}$  of the total order  $\leq_E$  to the interval  $[p, q]$  and define  $f_q : [p, q] \rightarrow [p, q]$  by letting  $f_q(r) = f(r)$  if  $r \in [p, q)$  and putting  $f_q(q) = q$ . Then  $\mathbb{I}_{[p, q]} = ([p, q], f_q, p)$  is the iterator given by Proposition 5.2 for the totally ordered set  $([p, q], \leq_{[p, q]})$  and by Proposition 5.2  $\mathbb{I}_{[p, q]}$  is minimal. But this is just the statement in Lemma 9.1.  $\square$

In what follows let  $X$  be an arbitrary non-empty set and let  $\bullet$  be a binary operation on  $X$ . If  $I$  is an interval then an  $I$ -tuple with values in  $X$  is just a mapping  $s \mapsto x_s$  from  $I$  to  $X$ . If the interval is given explicitly as  $[p, q]$  then a  $[p, q]$ -tuple will usually be written as  $(x_p, \dots, x_q)$ .

**Proposition 9.1** *Let  $[p, q]$  be an interval and let  $(x_p, \dots, x_q)$  be a  $[p, q]$ -tuple. Then there exists a unique  $[p, q]$ -tuple  $(\pi_p, \dots, \pi_q)$  with  $\pi_p = x_p$  and such that  $\pi_{f(s)} = \pi_s \bullet x_{f(s)}$  for all  $s \in [p, q]$ .*

*Proof* Let  $B$  be the subset of  $[p, q]$  consisting of those  $s$  for which there exists a unique  $[p, s]$ -tuple  $(\pi_p^s, \dots, \pi_s^s)$  with  $\pi_p^s = x_p$  and such that  $\pi_{f(y)}^s = \pi_y^s \bullet x_{f(y)}$  for all

$y \in [p, x)$ . Then  $p \in B$  with  $(x_p)$  the unique  $[p, p]$ -tuple. Thus let  $s \in B \setminus \{q\}$  and let  $(\pi_p^s, \dots, \pi_s^s)$  be the corresponding unique  $[p, s]$ -tuple. Then we can extend this  $[p, s]$ -tuple to a  $[p, f(s)]$ -tuple by putting  $\pi_y^{f(s)} = \pi_y^s$  for each  $y \in [p, s]$  and letting  $\pi_{f(s)}^{f(s)} = \pi(s^s \bullet x_{f(s)})$ . Then  $(\pi_p^{f(s)}, \dots, \pi_{f(s)}^{f(s)})$  is the unique  $[p, f(s)]$ -tuple having the required properties and hence  $f(s) \in B$ . Therefore by Lemma 9.1  $B = [p, q]$ .  $\square$

We denote the element  $\pi_q$  of  $X$  defined above by  $\bullet(x_p, \dots, x_q)$ .

Let  $(x_1, x_2, x_3, x_4)$  be a  $[1, 4]$ -tuple (with  $[1, 4] \subset \mathbb{N}$  and let  $(\pi_1, \pi_2, \pi_3, \pi_4)$  be the  $[1, 4]$ -tuple given by the above proposition. Then

$$\begin{aligned} \pi_1 &= x_1, \pi_2 = \pi_1 \bullet x_2 = x_1 \bullet x_2, \pi_3 = \pi_2 \bullet x_3 = (x_1 \bullet x_2) \bullet x_3, \\ \pi_4 &= \pi_3 \bullet x_4 = ((x_1 \bullet x_2) \bullet x_3) \bullet x_4. \text{ Hence } \bullet(x_1, x_2, x_3, x_4) = ((x_1 \bullet x_2) \bullet x_3) \bullet x_4. \end{aligned}$$

**Theorem 9.1** Let  $p, q, r \in E$  with  $p \leq q$  and  $f(q) \leq r$  and note that  $[p, r]$  is the disjoint union of  $[p, q]$  and  $[f(q), r]$ . Let  $(x_1, \dots, x_r)$  be a  $[p, r]$ -tuple and so we have a  $[p, q]$ -tuple  $(x_p, \dots, x_q)$  and a  $[f(q), r]$ -tuple  $(x_{f(q)}, \dots, x_r)$ . Suppose that the operation  $\bullet$  is associative. Then  $\bullet(x_p, \dots, x_r) = \alpha \bullet \beta$ , where  $\alpha = \bullet(x_p, \dots, x_q)$  and  $\beta = \bullet(x_{f(q)}, \dots, x_r)$ .

*Proof* For each  $s \in [f(q), r]$  put  $\beta_s = \bullet(x_{f(q)}, \dots, x_s)$  and let

$$B = \{s \in [f(q), r] : \bullet(x_p, \dots, x_s) = \alpha \bullet \beta_s\}.$$

Then  $f(q) \in B$  since  $\bullet(x_p, \dots, x_{f(q)}) = \bullet(x_p, \dots, x_q) \bullet x_{f(q)} = \alpha \bullet \beta_{f(q)}$ . Thus let  $s \in B \cap [f(q), r)$ . Then  $\bullet(x_p, \dots, s) = \alpha \bullet \beta_s$  and

$$\bullet(x_p, \dots, x_{f(s)}) = \bullet(x_p, \dots, x_s) \bullet x_{f(s)} = (\alpha \bullet \beta_s) \bullet x_{f(s)} = \alpha \bullet (\beta_s \bullet x_{f(s)}) = \alpha \bullet \beta_{f(s)}$$

and hence  $f(s) \in B$ . Thus by Lemma 9.1  $B = [f(q), r]$ , i.e.,  $\bullet(x_p, \dots, x_r) = \alpha \bullet \beta$ .  $\square$

If  $I$  is an interval then an interval partition of  $I$  is a partition  $\mathcal{Q}$  such that each element of  $\mathcal{Q}$  is also an interval. From now on partition always means interval partition. The partition consisting of the singleton sets  $[r, r]$ ,  $r \in I$ , will be denoted by  $\mathcal{Q}_0$  and the trivial partition  $\{I\}$  by  $\mathcal{Q}_T$ .

Let  $\mathcal{Q}$  be a partition of  $I$  and let  $J = [r, s]$  and  $K = [u, v]$  be elements of  $\mathcal{Q}$ . Then  $(J, K)$  will be called an adjacent pair if  $f(s) = u$ . In this case  $J \cup K = [r, v]$  is the disjoint union of  $J$  and  $K$ . A partition  $\mathcal{Q}'$  is said to be an  $I$ -reduction of  $\mathcal{Q}$  if there exists an adjacent pair  $(J, K)$  such that  $\mathcal{Q}' = \mathcal{Q} \setminus \{J, K\} \cup \{J \cup K\}$ . The partition  $\mathcal{Q}'$  will be denoted by  $\mathcal{Q}(J, K)$ . Consider the interval  $I = [p, q]$  to be fixed and let  $\mathcal{Q}$  be a subset of the set of all partitions of  $I$  containing  $\mathcal{Q}_0$  and  $\mathcal{Q}_T$ . Put



$\mathbb{Q}_S = \mathbb{Q} \setminus \{\mathbb{Q}_T\}$  and let  $\Phi : \mathbb{Q} \rightarrow \mathbb{Q}$  be a mapping with  $\Phi(\mathbb{Q}_T) = \mathbb{Q}_T$ . Thus there is the iterator  $\mathbb{R} = (\mathbb{Q}, \Phi, \mathbb{Q}_0)$  and  $\mathbb{R}$  will be called an *I-reduction* if it is minimal and  $\Phi(\mathbb{Q})$  is a reduction of  $\mathbb{Q}$  for each  $\mathbb{Q} \in \mathbb{Q}_S$ . Note that if  $p = q$  then there is a unique *I-reduction* with  $\mathbb{Q} = \mathbb{Q}_T\{I\}$  and with  $\Phi$  the identity mapping. If  $q = f(p)$  then there is also a unique *I-reduction* with  $\mathbb{Q} = \{\mathbb{Q}_0, \mathbb{Q}_T\}$  and with  $\Phi(\mathbb{Q}_0) = \mathbb{Q}_T$  and  $\Phi(\mathbb{Q}_T) = \mathbb{Q}_T$ . In what follows let  $\mathbb{R} = (\mathbb{Q}, \Phi, \mathbb{Q}_0)$  be an *I-reduction*. Thus  $\mathbb{R}$  is a finite minimal iterator with fixed-point  $\mathbb{Q}_T$  and so let  $\leq$  be the total order on  $\mathbb{Q}$  given in Proposition 5.1. Let  $<$ ,  $>$  and  $\geq$  have their usual meanings. If  $A$  and  $B$  are finite sets then as before  $A \preceq B$  means there is an injective mapping  $h : A \rightarrow B$  and we write  $A \prec B$  if both  $A \preceq B$  and  $A \not\approx B$  hold. Moreover if  $B \neq \emptyset$  then we write  $A \prec_0 B$  if  $A \approx B \setminus \{b\}$ , where  $b$  is any element in  $B$ . If  $\mathbb{Q} \in \mathbb{Q}_S$  then  $\Phi(\mathbb{Q}) \prec_0 \mathbb{Q}$ .

**Lemma 9.2** *If  $\mathbb{Q}, \mathbb{Q}' \in \mathbb{Q}$  with  $\mathbb{Q} \approx \mathbb{Q}'$  then  $\mathbb{Q} = \mathbb{Q}'$ .*

*Proof* Put  $\mathcal{S} = \{\mathbb{Q}_0\} \cup \{\mathbb{Q} \in \mathbb{Q} \setminus \{\mathbb{Q}_0\} : \mathbb{Q} \prec \mathbb{Q}_0\}$  and let  $\mathbb{Q} \in \mathcal{S} \setminus \{\mathbb{Q}_0\}$ . Then  $\mathbb{Q} \prec \mathbb{Q}_0$  and thus either  $\mathbb{Q} \in \mathbb{Q}_S$ , in which case  $\Phi(\mathbb{Q}) \prec \mathbb{Q}$  or  $\mathbb{Q} = \mathbb{Q}_T$ , in which case  $\Phi(\mathbb{Q}) = \mathbb{Q}$ . In both cases  $\Phi(\mathbb{Q}) \prec \mathbb{Q}_0$  and so  $\Phi(\mathbb{Q}) \in \mathcal{S}$ . Thus  $\mathcal{S}$  is  $\Phi$ -invariant and contains  $\mathbb{Q}_0$ . Hence  $\mathcal{S} = \mathbb{Q}$ , which shows that  $\mathbb{Q} \prec \mathbb{Q}_0$  and in particular  $\mathbb{Q} \not\approx \mathbb{Q}_0$  for all  $\mathbb{Q} \in \mathbb{Q} \setminus \{\mathbb{Q}_0\}$ . Now let  $\mathbb{Q}, \mathbb{Q}' \in \mathbb{Q}$  with  $\mathbb{Q} \neq \mathbb{Q}'$  and suppose that  $\mathbb{Q} \approx \mathbb{Q}'$ . We can assume that  $\mathbb{Q} < \mathbb{Q}'$  and that  $\mathbb{Q}$  is the least element of  $\mathbb{Q}$  such that  $\mathbb{Q} \approx \mathbb{Q}'$  for some  $\mathbb{Q}' > \mathbb{Q}$ . Then  $\mathbb{Q} \neq \mathbb{Q}_0$  since  $\mathbb{Q} \not\approx \mathbb{Q}_0$  for all  $\mathbb{Q} \in \mathbb{Q} \setminus \{\mathbb{Q}_0\}$  and so also  $\mathbb{Q}' \neq \mathbb{Q}_0$ . Hence by Proposition 3.3  $\mathbb{Q} = \Phi(\mathcal{U})$  and  $\mathbb{Q}' = \Phi(\mathcal{U}')$  for some  $\mathcal{U}, \mathcal{U}' \in \mathbb{Q}$ . Now  $\mathcal{U} \prec_0 \mathbb{Q}, \mathcal{U}' \prec_0 \mathbb{Q}'$  and  $\mathbb{Q} \approx \mathbb{Q}'$  and it follows from Proposition 5.1 (4) that  $\mathcal{U}' > \mathcal{U}$  and  $\mathcal{U} \approx \mathcal{U}'$ . But  $\mathcal{U} < \mathbb{Q}$ , which contradicts the minimality of  $\mathbb{Q}$ . Therefore if  $\mathbb{Q}, \mathbb{Q}' \in \mathbb{Q}$  with  $\mathbb{Q} \neq \mathbb{Q}'$  Then  $\mathbb{Q} \not\approx \mathbb{Q}'$  and hence if  $\mathbb{Q}, \mathbb{Q}' \in \mathbb{Q}$  with  $\mathbb{Q} \approx \mathbb{Q}'$  then  $\mathbb{Q} = \mathbb{Q}'$ .  $\square$

**Proposition 9.2** *For each  $J \in \mathcal{P}_0(I)$  there exists a unique  $\mathbb{Q} \in \mathbb{Q}$  with  $\mathbb{Q} \approx J$ .*

*Proof* Let  $\mathcal{S} = \{\emptyset\} \cup \{J \in \mathcal{P}_0(I) : \text{there exists } \mathbb{Q} \in \mathbb{Q} \text{ with } \mathbb{Q} \approx J\}$ . Let  $J \in \mathcal{S}^p$  and  $j \in I \setminus J$ ; put  $J' = J \cup \{j\}$ . If  $J = \emptyset$  and so  $J' = \{j\}$  then  $\mathbb{Q}_T \approx J'$ . If  $J \neq \emptyset$  then  $\mathbb{Q} \approx J$  for some  $\mathbb{Q} \in \mathbb{Q}$  and  $\mathbb{Q} \neq \mathbb{Q}_0$ , since  $\mathbb{Q}_0 \approx I$  and  $J$  is a proper subset of  $I$ . Thus by Proposition 3.3  $\mathbb{Q} = \Phi(\mathbb{Q}')$  for some  $\mathbb{Q}' \in \mathbb{Q}$  and then  $\mathbb{Q}' \approx J'$ , since  $\mathbb{Q} \prec_0 \mathbb{Q}'$ . Thus in both cases  $J' \in \mathcal{S}$  and therefore  $\mathcal{S} = \mathcal{P}(I)$ . This shows that for each  $J \in \mathcal{P}_0(I)$  there exists  $\mathbb{Q} \in \mathbb{Q}$  with  $\mathbb{Q} \approx J$ . The uniqueness follows from Lemma 9.2.  $\square$

**Lemma 9.3** *Let  $A$  and  $B$  be finite sets, let  $\mathcal{S} \subset \mathcal{P}(A)$  and suppose that for each  $J \in \mathcal{P}_0(B)$  there exists a unique  $C \in \mathcal{S}$  with  $C \approx J$ . Then  $B \approx \mathcal{S}$ . In particular, Proposition 9.2 implies that  $I \approx \mathbb{Q}$ .*

*Proof* There is a unique surjective mapping  $\alpha : \mathcal{P}_0(B) \rightarrow \mathcal{S}$  with  $\alpha(J) \approx J$  for each  $J \in \mathcal{P}_0(B)$ . Let  $\approx'$  be the restriction of the equivalence relation  $\approx$  to  $\mathcal{P}_0(B)$  and let  $\mathcal{E}(B)$  be the set of equivalence classes. Then the mapping  $\alpha$  induces a bijective mapping  $\alpha' : \mathcal{E}(B) \rightarrow \mathcal{S}$  and hence by Proposition 2.13  $B \approx \mathcal{S}$ .  $\square$

**Proposition 9.3** Suppose  $\mathbb{R} = (\mathbb{Q}, \Phi, \mathcal{Q}_0)$  satisfies the requirements for being an  $I$ -reduction except for the assumption that it be minimal. Then  $I \preceq \mathbb{Q}$  and  $I \approx \mathbb{Q}$  if and only if  $\mathbb{R}$  is minimal.

*Proof* Let  $\mathbb{Q}_0$  be the least  $\Phi$ -invariant subset of  $\mathbb{Q}$  containing  $\mathcal{Q}_0$  and let  $\Phi_0$  be the restriction of  $\Phi$  to  $\mathbb{Q}_0$ . Then  $\mathbb{R}_0 = (\mathbb{Q}_0, \Phi_0, \mathcal{Q}_0)$  is minimal and thus is an  $I$ -reduction and so by Lemma 9.3  $I \approx \mathbb{Q}_0$ . Hence  $I \preceq \mathbb{Q}$  and  $I \approx \mathbb{Q}$  if and only if  $\mathbb{R}$  is minimal.  $\square$

In what follows assume that  $\mathbb{R}$  is minimal.

**Proposition 9.4** Let  $s \mapsto x_s$  be an  $I$ -tuple. Then for each  $\mathcal{Q} \in \mathbb{Q}$  there is a unique mapping  $\omega_{\mathcal{Q}} : \mathcal{Q} \rightarrow X$  with  $\omega_{\mathcal{Q}_0}([z, z]) = x_z$  for each  $z \in I$  and such that if  $\mathcal{Q} \in \mathbb{Q}_S$  with  $\Phi(\mathcal{Q}) = \mathcal{Q}(J, K)$  then  $\omega_{\Phi(\mathcal{Q})}(M) = \omega_{\mathcal{Q}}(M)$  if  $M \neq J \cup K$  and  $\omega_{\Phi(\mathcal{Q})}(J \cup K) = \omega_{\mathcal{Q}}(J) \bullet \omega_{\mathcal{Q}}(K)$ .

*Proof* For each  $\mathcal{Q} \in \mathbb{Q}$  let  $P(\mathcal{Q})$  be the statement that for each  $\mathcal{U} \leq \mathcal{Q}$  there exists a unique mapping  $\omega_{\mathcal{U}}^{\mathcal{Q}} : \mathcal{U} \rightarrow X$  with  $\omega_{\mathcal{Q}_0}^{\mathcal{Q}}([z, z]) = x_z$  for each  $z \in I$  and such that if  $\mathcal{U} \in \mathbb{Q}_S$  with  $\Phi(\mathcal{U}) = \mathcal{U}(J, K)$  then  $\omega_{\Phi(\mathcal{U})}^{\mathcal{Q}}(M) = \omega_{\mathcal{U}}^{\mathcal{Q}}(M)$  if  $M \neq J \cup K$  and  $\omega_{\Phi(\mathcal{U})}^{\mathcal{Q}}(J \cup K) = \omega_{\mathcal{U}}^{\mathcal{Q}}(J) \bullet \omega_{\mathcal{U}}^{\mathcal{Q}}(K)$ . Then clearly  $P(\mathcal{Q}_0)$  holds and so let  $\mathcal{Q} \in \mathbb{Q}$  be such that  $P(\mathcal{Q})$  holds; put  $\mathcal{V} = \Phi(\mathcal{Q})$ . We must show that  $P(\mathcal{V})$  holds and if  $\mathcal{U} \leq \mathcal{V}$  then either  $\mathcal{U} \leq \mathcal{Q}$  or  $\mathcal{U} = \mathcal{V}$ . If  $\mathcal{U} \leq \mathcal{Q}$  then put  $\omega_{\mathcal{U}}^{\mathcal{V}} = \omega_{\mathcal{U}}^{\mathcal{Q}}$ . Finally, if  $\mathcal{U} = \mathcal{V} \in \mathbb{Q}_S$  with  $\Phi(\mathcal{V}) = \mathcal{V}(J, K)$  then put  $\omega_{\Phi(\mathcal{V})}^{\mathcal{V}}(M) = \omega_{\mathcal{Q}}^{\mathcal{Q}}(M)$  if  $M \neq J \cup K$  and  $\omega_{\Phi(\mathcal{V})}^{\mathcal{V}}(J \cup K) = \omega_{\mathcal{Q}}^{\mathcal{Q}}(J) \bullet \omega_{\mathcal{Q}}^{\mathcal{Q}}(K)$ . Then  $\{\omega_{\mathcal{U}}^{\mathcal{V}} : \mathcal{U} \leq \mathcal{V}\}$  are the unique mappings satisfying the requirements for  $P(\mathcal{V})$  and hence  $P(\mathcal{V})$  holds. Therefore  $P(\mathcal{Q})$  holds for all  $\mathcal{Q} \in \mathbb{Q}$ . We now define  $\omega_{\mathcal{Q}} = \omega_{\mathcal{Q}^T}^{\mathcal{Q}}$  for each  $\mathcal{Q} \in \mathbb{Q}$ .  $\square$

Note that  $\omega_{\mathcal{Q}_T}$  is a mapping from the singleton set  $\{I\}$  to  $X$  and this element of  $X$  will be denoted by  $\bullet_{\mathbb{R}}(s \mapsto x_s)$ . If  $p = q$  then  $\bullet_{\mathbb{R}}(x_p) = x_p$  and if  $q = f(p)$  then  $\bullet_{\mathbb{R}}(x_p, x_q) = x_p \bullet x_q$ , where in both cases  $\mathbb{R}$  is the unique  $I$ -reduction.

The element  $\bullet(s \mapsto x_s)$  of  $X$  arising from Proposition 9.1 is obtained using the  $I$ -reduction  $\mathbb{R} = (\mathbb{Q}, \Phi, \mathcal{Q}_0)$  where  $\Phi(\mathcal{Q}) = \mathcal{Q}(L_{\mathcal{Q}}^1, L_{\mathcal{Q}}^2)$  with  $(L_{\mathcal{Q}}^1, L_{\mathcal{Q}}^2)$  the adjacent pair consisting of the first and second elements in the partition  $\mathcal{Q}$ .

**Theorem 9.2** Suppose that  $\bullet$  is associative. Then

$$\bullet_{\mathbb{R}}(s \mapsto x_s) = \bullet(s \mapsto x_s)$$

for each  $I$ -tuple  $(s \mapsto x_s)$  and each reduction  $\mathbb{R}$ .

*Proof* In what follows we assume that  $I$  contains at least two elements and so by Theorem 5.1 (2) there exists a unique  $\mathcal{Q}_2 \in \mathcal{Q}$  with  $\Phi(\mathcal{Q}_2) = \mathcal{Q}_T$ . Clearly the partition  $\mathcal{Q}_2$  consists of exactly two components. Thus if  $I = [p, q]$  then there exists  $t \in [p, q]$  with  $p \leq_E t <_E q$  such that  $\mathcal{Q}_2 = \{L, R\}$ , where  $L = [p, t]$  and  $R = [f(t), q]$ .

**Lemma 9.4** For each  $\mathcal{Q} \in \mathcal{Q}_S$  and each  $Q \in \mathcal{Q}$  either  $Q \subset L$  or  $Q \subset R$ .

*Proof* Let  $\mathcal{Q}_0$  denote the set of those  $\mathcal{Q} \in \mathcal{Q}$  which contain an element  $Q$  which intersects both  $L$  and  $R$ . Then  $\mathcal{Q}_0$  is  $\Phi$ -invariant: Let  $\mathcal{Q} \in \mathcal{Q}$ ; then each component of  $\Phi(\mathcal{Q})$  is either equal to a component of  $\mathcal{Q}$  or is the union of two adjacent components of  $\mathcal{Q}$ . Thus if  $\mathcal{Q} \in \mathcal{Q}_0$  then  $\Phi(\mathcal{Q}) \in \mathcal{Q}_0$ . Note that  $\mathcal{Q}_2 \notin \mathcal{Q}_0$  but  $\mathcal{Q}_T \in \mathcal{Q}_0$ . Suppose  $\mathcal{Q}_0 \setminus \{\mathcal{Q}_T\}$  is non-empty; then by Proposition 2.14 it contains an element  $\mathcal{U}$  which is maximum with respect to the total order  $\leq$  and  $\mathcal{U} < \mathcal{Q}_2$ , since  $\mathcal{Q}_2 \notin \mathcal{Q}_0$ . But then  $\Phi(\mathcal{U}) \in \mathcal{Q}_0 \setminus \{\mathcal{Q}_T\}$ , which contradicts the maximality of  $\mathcal{U}$ . Therefore  $\mathcal{Q}_0 \setminus \{\mathcal{Q}_T\} = \emptyset$  and hence for each  $\mathcal{Q} \in \mathcal{Q}_S$  and each  $Q \in \mathcal{Q}$  either  $Q \subset L$  or  $Q \subset R$ .  $\square$

For each  $\mathcal{Q} \in \mathcal{Q}_S$  let  $\mathcal{Q}_L = \{Q \in \mathcal{Q} : Q \subset L\}$  and  $\mathcal{Q}_R = \{Q \in \mathcal{Q} : Q \subset R\}$ . Then  $\mathcal{Q}_L$  is a partition of  $L$  and  $\mathcal{Q}_R$  a partition of  $R$ . Moreover  $\mathcal{Q}_L$  and  $\mathcal{Q}_R$  are disjoint subsets of  $\mathcal{Q}$  whose union is  $\mathcal{Q}$ .

Put  $\mathcal{Q}_Z = \{\mathcal{Q} \in \mathcal{Q}_S : \Phi(\mathcal{Q}) \in \mathcal{Q}_S\}$  and so  $\mathcal{Q}_Z = \mathcal{Q}_S \setminus \{\mathcal{Q}_2\}$ . If  $\mathcal{Q} \in \mathcal{Q}_Z$  with  $\Phi(\mathcal{Q}) = \mathcal{Q}(J, K)$ ; then either  $J \cup K \subset L$  or  $J \cup K \subset R$ . Thus we can define a mapping  $\lambda : \mathcal{Q}_Z \rightarrow \{L, R\}$  by letting  $\lambda(\mathcal{Q}) = L$  if  $J \cup K \subset L$  and  $\lambda(\mathcal{Q}) = R$  if  $J \cup K \subset R$ .

Let  $\mathcal{Q} \in \mathcal{Q}_Z$  and so  $\Phi(\mathcal{Q})$  is a reduction of  $\mathcal{Q}$ . If  $\lambda(\mathcal{Q}) = L$  then  $\Phi(\mathcal{Q})_L \prec_0 \mathcal{Q}_L$  and  $\Phi(\mathcal{Q})_R = \mathcal{Q}_R$  and if  $\lambda(\mathcal{Q}) = R$  then  $\Phi(\mathcal{Q})_L = \mathcal{Q}_L$  and  $\Phi(\mathcal{Q})_R \prec_0 \mathcal{Q}_R$ .

Let  $\mathbb{U}$  be the set of  $L$ -partitions having the form  $\mathcal{Q}_L$  for some  $\mathcal{Q} \in \mathcal{Q}_S$ . Thus  $\mathcal{Q}_L \in \mathbb{U}$  for each  $\mathcal{Q} \in \mathcal{Q}_S$  but in general the representation as  $\mathcal{Q}_L$  cannot be unique. Note that  $\mathbb{U}$  contains the elements  $\mathcal{U}_0 = (\mathcal{Q}_0)_L$  and  $\mathcal{U}_T = (\mathcal{Q}_2)_L = \{L\}$ .

For each  $\mathcal{U} \in \mathbb{U}$  let  $\mathcal{Q}_{\mathcal{U}} = \{\mathcal{Q} \in \mathcal{Q}_S : \mathcal{Q}_L = \mathcal{U}\}$  and let  $\mathcal{U}^{\triangleleft}$  be the largest element in  $\mathcal{Q}_{\mathcal{U}}$ . Then  $\mathcal{U} = \mathcal{U}_L^{\triangleleft}$ , since  $\mathcal{U}^{\triangleleft} \in \mathcal{Q}_{\mathcal{U}}$ .

Define a mapping  $\Phi_L : \mathbb{U} \rightarrow \mathbb{U}$  by letting  $\Phi_L(\mathcal{U}) = \Phi(\mathcal{U}^\triangleleft)_L$  if  $\mathcal{U}^\triangleleft \in \mathbb{Q}_Z$  and letting  $\Phi_L(\mathcal{U}) = (\mathcal{U}_T)$  if  $\mathcal{U}^\triangleleft = \mathcal{Q}_2$ . In particular  $\Phi_L(\mathcal{U}_T) = \mathcal{U}_T$ .

We thus have the iterator  $\mathbb{R}_L = (\mathbb{U}, \Phi_L, (\mathcal{U}_0))$ .

Let  $\mathbb{U}_S = \mathbb{U} \setminus \{\mathcal{U}_T\}$ , thus  $\mathcal{U} \in \mathbb{U}_S$  if and only if  $\mathcal{U}^\triangleleft \in \mathbb{Q}_Z$ .

**Lemma 9.5** (1) Let  $\mathcal{U} \in \mathbb{U}_S$  and so  $\mathcal{Q} = \mathcal{U}^\triangleleft \in \mathbb{Q}_Z$ . Then  $\Phi(\mathcal{Q})$  is a reduction of  $\mathcal{Q}$  with  $\lambda(\mathcal{Q}) = L$  and  $\Phi_L(\mathcal{U})$  is an  $L$ -reduction of  $\mathcal{U}$ .

(2) Let  $\mathcal{Q} \in \mathbb{Q}_Z$  with  $\lambda(\mathcal{Q}) = L$ . Then  $\mathcal{Q} = (\mathcal{Q}_L)^\triangleleft$  and  $\Phi_L(\mathcal{Q}_L) = \Phi(\mathcal{Q})_L$ .

*Proof* (1) We have  $\Phi(\mathcal{U}^\triangleleft) \notin \mathbb{Q}_\mathcal{U}$  and so  $\Phi(\mathcal{U}^\triangleleft)_L \neq \mathcal{U} = \mathcal{U}_L^\triangleleft$ , i.e.,  $\Phi(\mathcal{U}^\triangleleft)_L \neq \mathcal{U}_L^\triangleleft$ . Thus  $\Phi(\mathcal{U}^\triangleleft) \neq \mathcal{U}^\triangleleft$  and hence  $\Phi(\mathcal{U}^\triangleleft)$  is a reduction of  $\mathcal{U}^\triangleleft$  with  $\lambda(\mathcal{U}^\triangleleft) = L$ . It follows that  $\Phi_L(\mathcal{U}) = \Phi(\mathcal{U}^\triangleleft)_L$  is an  $L$ -reduction of  $\mathcal{U}$ .

(2) We have  $\Phi(\mathcal{Q})$  is a reduction of  $\mathcal{Q}$  and  $\lambda(\mathcal{Q}) = L$  and so  $\Phi(\mathcal{Q})_L \neq \mathcal{Q}_L$ . Thus  $\mathcal{Q}$  is the largest element  $\mathcal{V} \in \mathbb{Q}_S$  such that  $\mathcal{V}_L = \mathcal{Q}_L$  and hence  $\mathcal{Q} = (\mathcal{Q}_L)^\triangleleft$ . It follows that  $\Phi_L(\mathcal{Q}_L) = \Phi((\mathcal{Q}_L)^\triangleleft) = \Phi(\mathcal{Q})_L$ .  $\square$

Let  $\mathbb{Q}_L = \{\mathcal{Q} \in \mathbb{Q}_Z : \mathcal{Q} = \mathcal{U}^\triangleleft \text{ for some } \mathcal{U} \in \mathbb{U}_S\}$ . If  $\mathcal{Q} = \mathcal{U}^\triangleleft$  then  $\mathcal{Q}_L = \mathcal{U}$  and so  $\mathcal{U}$  is uniquely determined by  $\mathcal{Q}$ . Thus there is a mapping  $\delta_L : \mathbb{Q}_L \rightarrow \mathbb{U}_S$  such that  $\delta_L(\mathcal{Q}) = \mathcal{U}$  whenever  $\mathcal{Q} = \mathcal{U}^\triangleleft$ . If  $\delta_L(\mathcal{Q}) = \delta_L(\mathcal{Q}')$  then  $\mathcal{Q} = \mathcal{Q}' = \mathcal{U}^\triangleleft$  for some  $\mathcal{U} \in \mathbb{Q}_L$  and hence  $\delta_L$  is injective. But  $\delta_L$  is clearly also surjective and therefore  $\delta_L$  is a bijection. The inverse of  $\delta_L$  is the bijective mapping  $\gamma_L : \mathbb{U}_S \rightarrow \mathbb{Q}_L$  given by  $\gamma_L(\mathcal{U}) = \mathcal{U}^\triangleleft$  for all  $\mathcal{U} \in \mathbb{U}_S$ .

**Lemma 9.6**  $\mathbb{Q}_L = \{\mathcal{Q} \in \mathbb{Q}_Z : \Phi(\mathcal{Q}) \text{ is a reduction of } \mathcal{Q} \text{ with } \lambda(\mathcal{Q}) = L\}$ .

*Proof* If  $\mathcal{Q} \in \mathbb{Q}_L$  then by Lemma 9.5 (1)  $\Phi(\mathcal{Q})$  is a reduction of  $\mathcal{Q}$  with  $\gamma(\mathcal{Q}) = L$ . Conversely, if  $\Phi(\mathcal{Q})$  is a reduction of  $\mathcal{Q}$  with  $\gamma(\mathcal{Q}) = L$  then by Lem's 9.5 (2)  $\mathcal{Q} = (\mathcal{Q}_L)^\triangleleft$  and so  $\mathcal{Q} \in \mathbb{Q}_L$ .  $\square$

Let  $\mathbb{V}$  be the set of  $R$ -partitions having the form  $\mathcal{Q}_R$  for some  $\mathcal{Q} \in \mathbb{Q}_S$ . Note that  $\mathbb{V}$  contains the elements  $\mathcal{V}_0 = (\mathcal{Q}_0)_R$  and  $\mathcal{V}_T = (\mathcal{Q}_2)_R = \{R\}$ .

For each  $\mathcal{V} \in \mathbb{V}$  let  $\mathbb{Q}_\mathcal{V} = \{\mathcal{Q} \in \mathbb{Q}_S : \mathcal{Q}_R = \mathcal{V}\}$  and let  $\mathcal{V}^\triangleleft$  be the largest element in  $\mathbb{Q}_\mathcal{V}$ . Then  $\mathcal{V} = \mathcal{V}_R^\triangleleft$ , since  $\mathcal{V}^\triangleleft \in \mathbb{Q}_\mathcal{V}$ .

Define a mapping  $\Phi_R : \mathbb{V} \rightarrow \mathbb{V}$  by letting  $\Phi_R(\mathcal{V}) = \Phi(\mathcal{V}^\triangleleft)_R$  if  $\mathcal{V}^\triangleleft \in \mathbb{Q}_Z$  and letting  $\Phi_R(\mathcal{V}) = (\mathcal{V}_T)$  if  $\mathcal{V}^\triangleleft = \mathcal{Q}_2$ . In particular  $\Phi_R(\mathcal{V}_T) = \mathcal{V}_T$ .

We thus have the iterator  $\mathbb{R}_R = (\mathbb{V}, \Phi_R, (\mathcal{V}_0))$ .

Let  $\mathbb{V}_S = \mathbb{V} \setminus \{\mathcal{V}_T\}$ , thus  $\mathcal{V} \in \mathbb{V}_S$  if and only if  $\mathcal{V}^\triangleleft \in \mathbb{Q}_Z$ .

**Lemma 9.7** (1) Let  $\mathcal{V} \in \mathbb{V}_S$  and so  $\mathcal{Q} = \mathcal{V}^\triangleleft \in \mathbb{Q}_Z$ . Then  $\Phi(\mathcal{Q})$  is a reduction of  $\mathcal{Q}$  with  $\lambda(\mathcal{Q}) = R$  and  $\Phi_R(\mathcal{V})$  is an  $R$ -reduction of  $\mathcal{V}$ .

(2) Let  $\mathcal{Q} \in \mathbb{Q}_Z$  with  $\lambda(\mathcal{Q}) = R$ . Then  $\mathcal{Q} = (\mathcal{Q}_R)^\triangleleft$  and  $\Phi_R(\mathcal{Q}_R) = \Phi(\mathcal{Q})_R$ .

*Proof* This is the same as the proof Lemma 9.5.  $\square$

Let  $\mathbb{Q}_R = \{\mathcal{Q} \in \mathbb{Q}_Z : \mathcal{Q} = \mathcal{V}^\triangleleft \text{ for some } \mathcal{V} \in \mathbb{V}_S\}$ . Then as above there is a bijective mapping  $\delta_R : \mathbb{Q}_R \rightarrow \mathbb{V}_S$  such that  $\delta_R(\mathcal{Q}) = \mathcal{V}$  whenever  $\mathcal{Q} = \mathcal{V}^\triangleleft$ . The inverse of  $\delta_R$  is the bijective mapping  $\gamma_R : \mathbb{V}_S \rightarrow \mathbb{Q}_R$  given by  $\gamma_R(\mathcal{V}) = \mathcal{V}^\triangleleft$  for all  $\mathcal{V} \in \mathbb{V}_S$ .

**Lemma 9.8**  $\mathbb{Q}_R = \{\mathcal{Q} \in \mathbb{Q}_Z : \Phi(\mathcal{Q}) \text{ is a reduction of } \mathcal{Q} \text{ with } \lambda(\mathcal{Q}) = R\}$ .

*Proof* This is the same as the proof of Lemma 9.6.  $\square$

**Lemma 9.9** The sets  $\mathbb{U}$  and  $\mathbb{V}$  are disjoint and  $\mathbb{U} \cup \mathbb{V} \approx \mathbb{Q}$ .

*Proof* It is clear that  $\mathbb{U}$  and  $\mathbb{V}$  are disjoint and by Lemmas 9.6 and 9.8 and (since there are bijections  $\gamma_L : \mathbb{U}_S \rightarrow \mathbb{Q}_L$  and  $\gamma_R : \mathbb{V}_S \rightarrow \mathbb{Q}_R$ ) it then follows that  $\mathbb{U}_S \cup \mathbb{V}_S \approx \mathbb{Q}_Z$ . Thus  $\mathbb{U} \cup \mathbb{V} \approx \mathbb{Q}$ , since  $\mathbb{U} = \mathbb{U}_S \cup \{\mathcal{U}_T\}$ ,  $\mathbb{V} = \mathbb{V}_S \cup \{\mathcal{V}_T\}$  and  $\mathbb{Q} = \mathbb{Q}_Z \cup \{\mathcal{Q}_2, \mathcal{Q}_T\}$ .  $\square$

**Lemma 9.10** The iterator  $\mathbb{R}_L$  is an  $L$ -reduction and  $\mathbb{R}_R$  is an  $R$ -reduction.

*Proof* By Lemma 9.5 (1)  $\Phi_L(\mathcal{U})$  is an  $L$ -reduction of  $\mathcal{U}$  for each  $\mathcal{U} \in \mathbb{U}_S$  and by Lemma 9.7 (1)  $\Phi_R(\mathcal{V})$  is an  $R$ -reduction of  $\mathcal{V}$  for each  $\mathcal{V} \in \mathbb{V}_S$ . It remains to show that the iterators  $\mathbb{R}_L$  and  $\mathbb{R}_R$  are minimal and by Proposition 9.3 this amounts to showing that  $\mathbb{U} \approx L$  and  $\mathbb{V} \approx R$ . By Proposition 9.3  $\mathbb{Q} \approx I$ ,  $L \preceq \mathbb{U}$  and  $R \preceq \mathbb{V}$ . Also by Lemma 9.9  $\mathbb{U}$  and  $\mathbb{V}$  are disjoint with  $\mathbb{U} \cup \mathbb{V} \approx \mathbb{Q}$ , and of course  $L$  and  $R$  are disjoint with  $L \cup R = I$ . Suppose  $L \prec \mathbb{U}$ ; then  $I = L \cup R \prec \mathbb{U} \cup \mathbb{V} \approx \mathbb{Q}$  and this contradiction implies that  $\mathbb{U} \approx L$ . In the same way  $\mathbb{V} \approx R$ .  $\square$

If  $\mathcal{U}$  is an  $L$ -partition and  $\mathcal{V}$  an  $R$ -partition then the  $I$ -partition  $\mathcal{Q}$  with  $\mathcal{Q}_L = \mathcal{U}$  and  $\mathcal{Q}_R = \mathcal{V}$  will be denoted by  $\mathcal{U} + \mathcal{V}$ . If  $\mathcal{Q} \in \mathbb{Q}_S$  then  $\mathcal{Q} = \mathcal{Q}_L + \mathcal{Q}_R$  and  $\mathcal{Q}_L \in \mathbb{U}$  and  $\mathcal{Q}_R \in \mathbb{V}$ . Let  $\mathcal{Q}$  be an  $I$ -partition with  $\mathcal{Q} = \mathcal{U} + \mathcal{V}$  and let  $f_L : \mathcal{U} \rightarrow X$  and  $f_R : \mathcal{V} \rightarrow X$  be mappings. Then there is a mapping  $f_L \oplus f_R : \mathcal{Q} \rightarrow X$  defined by

$$(f_L \oplus f_R)(J) = \begin{cases} f_L(J) & \text{if } J \in \mathcal{U}, \\ f_R(J) & \text{if } J \in \mathcal{V}. \end{cases}$$

Consider the mappings given by Proposition 9.4 for  $\mathbb{Q}$ ,  $\mathbb{U}$  and  $\mathbb{V}$ . Thus for each  $\mathcal{Q} \in \mathbb{Q}$  there is the mapping  $\omega_{\mathcal{Q}} : \mathcal{Q} \rightarrow X$ , for each  $\mathcal{U} \in \mathbb{U}$  there is the mapping  $\omega_{\mathcal{U}}^L : \mathcal{U} \rightarrow X$  and for each  $\mathcal{V} \in \mathbb{V}$  there is the mapping  $\omega_{\mathcal{V}}^R : \mathcal{V} \rightarrow X$ .

**Lemma 9.11** *Let  $(x_p, \dots, x_q)$  be a  $[p, q]$ -tuple. Then  $\omega_{\mathcal{Q}} = \omega_{\mathcal{Q}_L}^L \oplus \omega_{\mathcal{Q}_R}^R$  for all  $\mathcal{Q} \in \mathbb{Q}_S$  and  $\omega_{\mathcal{Q}_T} = \omega_{\{L\}}^L \bullet \omega_{\{R\}}^R$ . Thus*

$$\bullet_{\mathbb{R}}(x_p, \dots, x_q) = (\bullet_{\mathbb{R}_L}(x_p, \dots, x_t)) \bullet (\bullet_{\mathbb{R}_R}(x_{f(t)}, \dots, x_q)).$$

*Proof* Let  $\mathbb{Q}' = \{\mathcal{Q} \in \mathbb{Q}_S : \omega_{\mathcal{Q}} = \omega_{\mathcal{Q}_L}^L \oplus \omega_{\mathcal{Q}_R}^R\}$  and so  $\mathcal{Q}_0 \in \mathbb{Q}'$ . Let  $\mathcal{Q} \in \mathbb{Q}' \cap \mathbb{Q}_Z$ . If  $\lambda(\mathcal{Q}) = L$  then by Lemma 9.5 (2)  $\Phi(\mathcal{Q}) = \Phi(\mathcal{Q})_L + \Phi(\mathcal{Q})_R = \Phi_L(\mathcal{Q}_L) + \mathcal{Q}_R$  and thus  $\omega_{\Phi(\mathcal{Q})} = \omega_{\Phi_L(\mathcal{Q}_L)}^L \oplus \omega_{\mathcal{Q}_R}^R = \omega_{\Phi(\mathcal{Q})_L}^L \oplus \omega_{\Phi(\mathcal{Q})_R}^R$ . In exactly the same way, if  $\lambda(\mathcal{Q}) = R$  then by Lemma 9.7 (2)  $\Phi(\mathcal{Q}) = \Phi(\mathcal{Q})_L + \Phi(\mathcal{Q})_R = \mathcal{Q}_L + \Phi_R(\mathcal{Q}_R)$  and hence  $\omega_{\Phi(\mathcal{Q})} = \omega_{\mathcal{Q}_L}^L \oplus \omega_{\Phi_R(\mathcal{Q}_R)}^R = \omega_{\Phi(\mathcal{Q})_L}^L \oplus \omega_{\Phi(\mathcal{Q})_R}^R$ . This shows that  $\Phi(\mathcal{Q}) \in \mathbb{Q}'$ . Thus  $\mathbb{Q}'$  is a  $\Phi$ -invariant subset of  $\mathbb{Q}$  containing  $\mathcal{Q}_0$  and so  $\mathbb{Q}' = \mathbb{Q}_S$ . Therefore  $\omega_{\mathcal{Q}} = \omega_{\mathcal{Q}_L}^L \oplus \omega_{\mathcal{Q}_R}^R$  for all  $\mathcal{Q} \in \mathbb{Q}_S$ . Finally,  $\mathcal{Q}_T = \Phi(\mathcal{Q}_2) = \Phi(\{L\} + \{R\})$  and it follows that  $\omega_{\mathcal{Q}_T} = \omega_{\Phi(\mathcal{Q}_2)} = \omega_{\{L\}}^L \bullet \omega_{\{R\}}^R$ .  $\square$

*Proof of Theorem 9.2:* If  $B$  is any non-empty finite set then by Lemma 2.6 there exists an  $I$ -reduction with  $I \approx B$ . For each non-empty finite set  $B$  let  $\mathbf{P}(B)$  be the statement that  $\bullet_{\mathbb{R}}(x_p, \dots, x_q) = \bullet(x_p, \dots, x_q)$  holds for each  $I$ -tuple  $(x_p, \dots, x_q)$  whenever  $\mathbb{R}$  is an  $I$ -reduction with  $I \approx B$ . Note that if  $B_1 \approx B_2$  then  $\mathbf{P}(B_1)$  holds if and only if  $\mathbf{P}(B_2)$  holds. Suppose there exists a non-empty finite set  $B$  for which  $\mathbf{P}(B)$  does not hold and let  $\mathcal{S}$  denote the set of non-empty subsets  $C$  of  $B$  for which  $\mathbf{P}(C)$  does not hold. Then  $\mathcal{S}$  is non-empty and so by Proposition 1.3 it contains a minimal element  $D$  and  $D$  contains at least two elements, since  $\mathbf{P}(\{a\})$  holds for each element  $a$ , because if  $I = \{a\}$  then there is only one  $I$ -reduction. If  $D'$  is any non-empty finite set with  $D' \prec D$  then  $\mathbf{P}(D')$  holds.

Now let  $I = [p, q]$  with  $I \approx D$ . Since  $\mathbf{P}(D)$  does not hold there exists an  $I$ -reduction  $\mathbb{R}$  and a  $[p, q]$ -tuple  $(x_p, \dots, x_q)$  such that  $\bullet_{\mathbb{R}}(x_p, \dots, x_q) \neq \bullet(x_p, \dots, x_q)$ . Then by Lemma 9.11  $\bullet_{\mathbb{R}}(x_p, \dots, x_q) = (\bullet_{\mathbb{R}_L}(x_p, \dots, x_t)) \bullet (\bullet_{\mathbb{R}_R}(x_{t+1}, \dots, x_q))$  and  $L \prec D$ ,  $R \prec D$ . Thus  $\mathbf{P}(L)$  and  $\mathbf{P}(R)$  hold and so

$$\bullet_{\mathbb{R}}(x_p, \dots, x_q) = (\bullet(x_p, \dots, x_t)) \bullet (\bullet(x_{t+1}, \dots, x_q)).$$

Hence by Theorem 9.1  $\bullet_{\mathbb{R}}(x_p, \dots, x_q) = \bullet(x_p, \dots, x_q)$ , and this contradiction shows that  $\mathbf{P}(B)$  holds for each non-empty finite set  $B$ . Therefore

$$\bullet_{\mathbb{R}}(x_p, \dots, x_q) = \bullet(x_p, \dots, x_q)$$

for each  $I$ -tuple  $(x_p, \dots, x_q)$  and each  $I$ -reduction  $\mathbb{R}$ .  $\square$

## 10 Binomial coefficients

For each finite set  $A$  and each  $B \subset A$  denote the set  $\{C \in \mathcal{P}(A) : C \approx B\}$  by  $A \Delta B$  and so  $A \Delta B \in \mathcal{P}(\mathcal{P}(A))$ . The set  $A \Delta B$  plays the role of a binomial coefficient: If  $|A| = n$  (with  $|A|$  the usual cardinality of the set  $A$ ) and  $|B| = k$  then  $|A \Delta B| = \binom{n}{k}$ . In this section we establish results which correspond to some of the usual identities for binomial coefficients.

If  $A$ ,  $B$  and  $C$  are finite sets then we write  $C \approx A \amalg B$  if there exist disjoint sets  $A'$  and  $B'$  with  $A \approx A'$ ,  $B \approx B'$  and  $C \approx A' \cup B'$ . (It is clear that whether this is the case does not depend on the choice of  $A'$  and  $B'$ .)

The following theorem corresponds to the identity

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

which is used to generate Pascal's triangle.

**Theorem 10.1** Let  $A$  be a finite set and  $B$  be a proper subset of  $A$ . Let  $a$  be an element not in  $A$  and let  $b \in A \setminus B$ . Then

$$(A \cup \{a\}) \Delta (B \cup \{a\}) \approx (A \Delta B) \amalg (A \Delta (B \cup \{b\})).$$

*Proof* Put  $D = (A \cup \{a\}) \Delta (B \cup \{a\})$ ,  $D_a = \{C \in (A \cup \{a\}) \Delta (B \cup \{a\}) : a \in C\}$  and  $D_b = \{C \in (A \cup \{a\}) \Delta (B \cup \{a\}) : a \notin C\}$ , so  $D$  is the disjoint union of  $D_a$  and  $D_b$ . Let  $C \in (A \cup \{a\}) \Delta (B \cup \{a\})$ . Then  $C \subset A \cup \{a\}$  with  $C \approx B \cup \{a\}$ . If  $C \in D_a$  then  $C' = C \setminus \{a\} \subset A$  and  $C' \approx B$  and in this case  $C' \in A \Delta B$ . If  $C \in D_b$  then  $C \subset A$  and  $C \approx B \cup \{b\}$  and in this case  $C \in A \Delta (B \cup \{b\})$ . There is thus a mapping  $\alpha : D_a \rightarrow A \Delta B$  given by  $\alpha(C) = C \setminus \{a\}$  and a mapping  $\beta : D_b \rightarrow A \Delta (B \cup \{b\})$  given by  $\beta(C) = C$ . Let  $C \in A \Delta B$ . Then  $C \subset A$  with  $C \approx B$  and so  $C \cup \{a\} \subset A \cup \{a\}$  with  $C \cup \{a\} \approx B \cup \{a\}$ . Hence  $C \cup \{a\} \in D_a$  and therefore there is a mapping  $\alpha' : A \Delta B \rightarrow D_a$  given by  $\alpha'(C) = C \cup \{a\}$ . Now let  $C \in A \Delta (B \cup \{b\})$ . Then  $C \subset A$  with  $C \approx (B \cup \{b\})$  and so  $C \cup \{a\} \subset A \cup \{a\}$  with  $C \cup \{a\} \approx (B \cup \{b\}) \cup \{a\}$ . Hence  $C \cup \{a\} \in D_b$  and therefore there is a mapping  $\beta' : A \Delta (B \cup \{b\}) \rightarrow D_b$  given by  $\beta'(C) = C \cup \{a\}$ .

It follows immediately that  $\alpha'$  is the inverse of  $\alpha$  and that  $\beta'$  is the inverse of  $\beta$ . Thus the mappings  $\alpha$  and  $\beta$  are both bijections. Therefore  $D_a \approx A \Delta B$  and  $D_b \approx A \Delta (B \cup \{b\})$  which shows that

$$(A \cup \{a\}) \Delta (B \cup \{a\}) = D = D_a \cup D_b \approx (A \Delta B) \amalg (A \Delta (B \cup \{b\})). \quad \square$$

If  $C$  is a finite set then, as in Section 8, let  $S_C$  denote the set(group) of bijections  $h : C \rightarrow C$ . If  $B$  is a subset of a finite set  $A$  then let  $I_{B,A}$  denote the set of injective mappings  $k : B \rightarrow A$ . Note that  $I_{A,A} = S_A$ ,  $I_{\emptyset,A} = \{\emptyset\}$  and  $I_{\{a\},A} \approx A$  for each  $a \in A$ .

**Theorem 10.2** *Let  $B$  be a subset of a finite set  $A$ . Then  $I_{B,A} \approx (A \Delta B) \times S_B$ .*

*Proof* Let  $u : I_{B,A} \rightarrow (A \Delta B)$  be the mapping with  $u(k) = k(B)$ . Then  $u$  is surjective and so by Lemma 2.5 there exists a mapping  $v : (A \Delta B) \rightarrow I_{B,A}$  with  $u(v(C)) = C$  for all  $C \in (A \Delta B)$  (and  $v$  is injective).

Let  $k \in I_{B,A}$ ; then  $k(B) \in A \Delta B$  and therefore there exists a bijective mapping  $t_k : B \rightarrow k(B)$ . (Note that  $t_k$  is not unique unless  $B$  is empty or contains only one element.) If  $k_1, k_2 \in I_{B,A}$  with  $k_1(B) = k_2(B)$  and  $h = (t_{k_2})^{-1} \circ t_{k_1}$  then  $h \in S_B$  and  $k_2 \circ h = k_1$ . On the other hand, if there exists  $h \in S_B$  with  $k_2 \circ h = k_1$  then  $k_1(B) = k_2(B)$ . Therefore  $k_1(B) = k_2(B)$  if and only if there exists  $h \in S_B$  such that  $k_2 \circ h = k_1$ . Note that if there exists  $h \in S_B$  such that  $k_2 \circ h = k_1$  then  $h$  is unique.

Now if  $C \in A \Delta B$  then  $v(C)$  is the unique element in  $I_{B,A}$  with  $v(C)(B) = C$ . In particular, it follows for each  $k \in I_{B,A}$  that  $v(k(B))$  is the unique element of  $I_{B,A}$  with  $v(k(B))(B) = k(B)$ . For each  $k \in I_{B,A}$  there thus exists a unique element  $s_k \in S_B$  such that  $k = v(k(B)) \circ s_k$ . Define  $G : I_{B,A} \rightarrow (A \Delta B) \times S_B$  by letting  $G(k) = (k(B), s_k)$ . Let  $(C, h) \in (A \Delta B) \times S_B$  and put  $k = v(C) \circ h$ . Then  $k \in I_{B,A}$  with  $k(B) = (v(C) \circ h)(B) = (v(C)(B) = C$  and  $s_k = h$ , since  $s_k$  is uniquely determined by the requirement that  $k = v(k(B)) \circ s_k$ . This shows that  $G$  is surjective. Now let  $j, k \in I_{B,A}$  with  $G(j) = G(k)$ . Then  $j(B) = k(B)$  and  $s_j = s_k$  and hence  $j = v(j(B)) \circ s_j = v(k(B)) \circ s_k = k$ . This shows that  $G$  is injective and therefore  $G$  is a bijection, i.e.,  $I_{B,A} \approx (A \Delta B) \times S_B$ .  $\square$

**Proposition 10.1** *Let  $B$  be a non-empty subset of a finite set  $A$ , let  $b \in B$  and put  $B' = B \cup \{b\}$ . Then  $I_{B',A} \approx I_{B,A} \times (A \setminus B)$ .*

*Proof* Let  $r : I_{B',A} \rightarrow I_{B,A}$  be the restriction mapping. Then  $r$  is surjective, since  $B \neq A$ . For each  $j \in I_{B,A}$

let  $p(j) = \{k \in I_{B',A} : r(k) = j\}$ . Now  $j(B) \approx B$  and for each  $c \in A \setminus j(B)$  there is a unique  $k \in p(j)$  with  $k(b) = c$ . Hence  $p(j) \approx A \setminus B$  and so let  $s_j : p(j) \rightarrow A \setminus B$  be a bijection. Define  $t : I_{B',A} \rightarrow (A \setminus B)$  by  $t(k) = s_j(k(b))$ , where  $j = r(k)$ . Now define  $R : I_{B',A} \rightarrow I_{B,A} \times (A \setminus B)$  by  $R(k) = (r(k), t(k))$  for each  $k \in I_{B',A}$ . Let  $k_1, k_2 \in I_{B',A}$  with  $R(k_1) = R(k_2)$ . Then  $r(k_1) = r(k_2)$  and  $t(k_1) = t(k_2)$ . Thus



$p(r(k_1)) = p(r(k_2))$  and  $s_j((k_1(b)) = s_j(k_2(b))$ , where  $j = r(k_1) = r(k_2)$ . Hence  $k_1(b) = k_2(b)$  and so  $k_1 = k_2$ , i.e.,  $R$  is injective. Now let  $(j, C) \in I_{B,A} \times (A \setminus B)$  and let  $k = (s_j)^{-1}(C)$ . It follows that  $R(k) = (r(k), t(k)) = (j, C)$  and so  $R$  is surjective. Therefore  $R$  is a bijection and hence  $I_{B'A} \approx I_{B,A} \times (A \setminus B)$ .  $\square$

The choice of the bijections  $s_j$ ,  $j \in I_{B,A}$  in the above proof can be made more explicit with help of Lemma 2.5: Let  $\Delta$  be the set of all subsets  $C$  of  $I_{B',A}$  with  $C \approx A \setminus B$  and let  $\Lambda$  be the set of all bijections  $q : C \rightarrow A \setminus B$  with  $C \in \Delta$ . Define  $u : \Lambda \rightarrow \Delta$  by letting  $u(q)$  be the domain of  $q$ . Then  $u$  is surjective and so by Lemma 2.5 there exists a mapping  $v : \Delta \rightarrow \Lambda$  with  $u \circ v = \text{id}_\Delta$ . For each  $j \in I_{B,A}$  put  $s_j = v(p(j))$ .

**Proposition 10.2** *Let  $A$  be a finite set, let  $a \notin A$  and put  $A' = A \cup \{a\}$ . Then  $S_{A'} \approx S_A \times A'$ .*

*Proof* Let  $p \in S_{A'}$  and suppose that  $p(a) \neq a$ . Then there exists a unique element  $c_p \in A$  with  $p(c_p) = a$ . Define  $\lambda[p] : A \rightarrow A$  by letting  $\lambda[p](d) = p(d)$  if  $p(d) \in A$  and  $\lambda[p](c_p) = p(a)$ , and thus  $\lambda[p] \in S_A$ . If  $p(a) = a$  then let  $\lambda[p]$  be the restriction of  $p$  to  $A$ . Define  $\Lambda : S_{A'} \rightarrow S_A \times A'$  by putting  $\Lambda(p) = (\lambda[p], p(a))$  for each  $p \in S_{A'}$ .

Let  $p, q \in S_{A'}$  with  $\Lambda(p) = \Lambda(q)$ . Then  $\lambda[p] = \lambda[q]$  and  $p(a) = q(a)$ . Assume first that  $b = p(a) \neq a$  and put  $r = \lambda[p]$ . Then  $r(c_p)$  and  $r(c_q)$  are both equal to  $b$  and hence  $c_p = c_q$  since  $r$  is a bijection. But if  $d \neq c_p$  then  $r(d) = p(d) = q(d)$  and it follows that  $p = q$ . If  $p(a) = a$  then  $\lambda[p]$  is the restriction of  $p$  to  $A$  and  $\lambda[q]$  is the restriction of  $q$  to  $A$  and it again follows that  $p = q$ . This shows that  $\Lambda$  is injective.

Now let  $q \in S_A$  and  $b \in A$ ; then there exists a unique  $b_q \in A$  with  $q(b_q) = b$ . Define  $\omega[q, b] : A' \rightarrow A'$  by  $\omega[q, b](c) = q(c)$  if  $c \in A \setminus \{b_q\}$ ,  $\omega[q, b](b_q) = a$  and  $\omega[q, b](a) = b$ , so  $\omega[q, b] \in S_{A'}$ . Then

$$(\Lambda(\omega[q, b]) = (\lambda[\omega[q, b]], \omega[q, b](a)) = (\lambda[\omega[q, b]], b) = (\lambda[r], b) = (s, b),$$

where  $r = \omega[q, b]$  and  $s = \lambda[r]$ . Thus  $r(c) = q(c)$  if  $c \in A \setminus \{b_q\}$ ,  $r(b_q) = a$  and  $r(a) = b$  (and where  $q(b_q) = b$ ). Also  $s(d) = r(d)$  if  $s(d) \in A \setminus \{c_r\}$  and  $s(c_r) = r(a)$  (and where  $r(c_r) = a$ ). Since  $r(c_a) = r(b_q) = a$  and  $r$  is a bijection it follows that  $b_q = c_r$ . Therefore  $s(d) = r(d) = q(d)$  for all  $d \in A \setminus \{c_r\}$  and  $s(c_r) = r(a) = b = q(b_q) = q(c_r)$  and hence  $s = q$ , i.e.,  $\Lambda(\omega[q, b]) = (q, b)$ . Moreover, it is clear that  $\Lambda(q') = (q, a)$  for each  $q \in S_A$ , where  $q'$  is the extension of  $q$  to  $S_{A'}$  with  $q'(a) = a$ . This shows that  $\Lambda$  is surjective and hence it is a bijection. In particular,  $S_{A'} \approx S_A \times A'$ .  $\square$

**Lemma 10.1** *Let  $B$  be a non-empty subset of a finite set  $A$ , let  $b \in B$  and put  $B' = B \cup \{b\}$ . Then  $I_{B',A} \times S_{A \setminus B'} \approx I_{B,A} \times S_{A \setminus B}$ .*

*Proof* By Proposition 10.1  $I_{B',A} \approx I_{B,A} \times (A \setminus B)$  and by Proposition 10.2 it follows that  $S_{A \setminus B} \approx S_{A \setminus B'} \times (A \setminus B)$ . Therefore

$$I_{B',A} \times S_{A \setminus B'} \approx I_{B,A} \times (A \setminus B) \times S_{A \setminus B'} \approx I_{B,A} \times S_{A \setminus B} . \quad \square$$

**Theorem 10.3** *Let  $B$  be a subset of a finite set  $A$ . Then  $S_A \approx I_{B,A} \times S_{A \setminus B}$ .*

*Proof* Let  $\mathcal{S}$  denote the set of subsets  $B$  of  $A$  for which  $S_A \approx I_{B,A} \times S_{A \setminus B}$ . Then  $\emptyset \in \mathcal{S}$ , since  $I_{\emptyset,A} = \{\emptyset\}$ . Let  $B \in \mathcal{S}$  and  $b \in A \setminus B$ ; put  $B' = B \cup \{b\}$ . Then  $S_A \approx I_{B,A} \times S_{A \setminus B}$  and so by Lemma 10.1  $S_A \approx I_{B',A} \times S_{A \setminus B'}$ . Hence  $B' \in \mathcal{S}$ , which shows that  $\mathcal{S}$  is an inductive  $A$ -system. Therefore  $\mathcal{S} = \mathcal{P}(A)$  and in particular  $A \in \mathcal{S}$ , i.e.,  $S_A \approx I_{B,A} \times S_{A \setminus B}$ .  $\square$

If  $B$  is a subset of a finite set  $A$  then by Theorem 10.3  $S_A \approx I_{B,A} \times S_{A \setminus B}$  and so there exists a bijective mapping  $h : S_A \rightarrow I_{B,A} \times S_{A \setminus B}$ . However, there does not seem to be a natural candidate for the mapping  $h$ .

The following theorem corresponds to the usual expression for binomial coefficients:

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}$$

**Theorem 10.4** *Let  $B$  be a subset of a finite set  $A$ . Then*

$$S_A \times (A \Delta B) \approx S_B \times S_{A \setminus B} .$$

*Proof* By Theorem 10.2  $I_{B,A} \approx (A \Delta B) \times S_B$  by Theorem 10.3  $S_A \approx I_{B,A} \times S_{A \setminus B}$ . Therefore  $S_A \approx I_{B,A} \times S_{A \setminus B} \approx (A \Delta B) \times S_B \times S_{A \setminus B}$ .  $\square$

The remark following Theorem 10.3 also applies here: There does not seem to be a natural candidate for a bijective mapping  $h : S_A \times (A \Delta B) \rightarrow S_B \times S_{A \setminus B}$ .

Theorem 10.5 below corresponds to the following well-known identity for binomial coefficients

$$\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$$

which holds for all  $0 \leq k \leq m \leq n$ .

**Theorem 10.5** *Let  $A$ ,  $B$  and  $C$  be finite sets with  $C \subset B \subset A$ . Then*

$$(A \Delta B) \times (B \Delta C) \approx (A \Delta C) \times ((A \setminus C) \Delta (B \setminus C)) .$$

*Proof* By Theorem 10.4 we have

$$S_A \times S_B \times (A \Delta B) \times (B \Delta C) \approx S_B \times S_{A \setminus B} \times S_C \times S_{B \setminus C}$$

$$\text{and } S_A \times S_{A \setminus C} \times (A \Delta C) \times ((A \setminus C) \Delta (B \setminus C)) \approx S_C \times S_{A \setminus C} \times S_{B \setminus C} \times S_{A \setminus B}$$

and therefore by Proposition 2.11 (cancellation law for finite sets)

$$S_A \times (A \Delta B) \times (B \Delta C) \approx S_{A \setminus B} \times S_C \times S_{B \setminus C} \approx S_C \times S_{B \setminus C} \times S_{A \setminus B}$$

$$\text{and } S_A \times (A \Delta C) \times ((A \setminus C) \Delta (B \setminus C)) \approx S_C \times S_{B \setminus C} \times S_{A \setminus B}$$

and hence

$$S_A \times (A \Delta B) \times (B \Delta C) \approx S_A \times (A \Delta C) \times ((A \setminus C) \Delta (B \setminus C)) .$$

Again making use of Proposition 2.11 it follows that

$$(A \Delta B) \times (B \Delta C) \approx (A \Delta C) \times ((A \setminus C) \Delta (B \setminus C)) . \quad \square$$

Theorem 10.5 implies there is a bijective mapping

$$h : (A \Delta B) \times (B \Delta C) \rightarrow (A \Delta C) \times ((A \setminus C) \Delta (B \setminus C))$$

but once again there does not seem to be a natural candidate for this mapping. It is worth noting that in many text-books the following simple combinatorial argument is often used to justify the identity

$$\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k} .$$

The left-hand side is the number of ways of first choosing  $m$  objects from a set of  $n$  objects and then choosing from these  $m$  objects a subset of  $k$  objects. But this is the same as first choosing  $k$  objects from the set of  $n$  objects and then choosing  $m-k$  objects from the remaining  $n-k$  objects, which is the right-hand side.

The combinatorial argument would seem to suggest how a bijective mapping  $h$  could be defined but it is not clear how to implement this.

## 11 Dilworth's decomposition theorem

In this section we prove Dilworth's decomposition theorem [2] by modifying a proof due to Galvin [3] to work with the present treatment of finite sets.

It is well-known that Dilworth's theorem can be used to provide straightforward proofs of further important combinatorial results such as the theorems of König, Menger, König-Egerváry and Hall. (See, for example [7].)

Recall that the set of non-empty subsets of a set  $E$  will be denoted by  $\mathcal{P}_0(E)$  and that by a partition of  $E$  we mean a subset  $\mathcal{Q}$  of  $\mathcal{P}_0(E)$  such that for each  $e \in E$  there exists a unique  $Q \in \mathcal{Q}$  such that  $e \in Q$ . In particular, different elements in a partition have to be disjoint. The only partition of the empty set  $\emptyset$  is the empty set  $\emptyset = \mathcal{P}_0(\emptyset)$ . If  $A$  is finite then by Propositions 2.3 and 1.1 any partition of  $A$  is also finite. To each partition  $\mathcal{Q}$  of a set  $E$  there is the evaluation map  $i_{\mathcal{Q}} : E \rightarrow \mathcal{Q}$ , where  $i_{\mathcal{Q}}(e)$  is the unique element in  $\mathcal{Q}$  containing  $e$ . If  $D \subset E$  then the restriction of  $i_{\mathcal{Q}}$  to  $D$  will be denoted by  $i_{\mathcal{Q}}^D$ .

Recall that a partial order on a set  $E$  is a mapping  $\leq : E \times E \rightarrow \mathbb{B}$  such that  $e \leq e$  for all  $e \in E$ ,  $e_1 \leq e_2$  and  $e_2 \leq e_1$  both hold if and only if  $e_1 = e_2$ , and  $e_1 \leq e_3$  holds whenever  $e_1 \leq e_2$  and  $e_2 \leq e_3$  for some  $e_2 \in E$ , and where as usual  $e_1 \leq e_2$  is written instead of  $\leq(e_1, e_2) = \top$ . A partially ordered set (or poset) is a pair  $(E, \leq)$  consisting of a set  $E$  and a partial order  $\leq$  on  $E$ . A finite poset  $(A, \leq)$  is a poset  $(A, \leq)$  with  $A$  a finite set.

If  $(E, \leq)$  is a poset and  $D$  a non-empty subset of  $E$  then  $d \in D$  is said to be a maximal element of  $D$  if  $d$  itself is the only element  $e \in D$  with  $d \leq e$ . By Proposition 2.14 every non-empty finite subset of  $E$  possesses a maximal element.

Let  $(E, \leq)$  be a poset. A subset  $C$  of  $E$  is called a chain if any two elements in  $C$  are comparable, i.e., if  $c \leq c'$  or  $c' \leq c$  for all  $c, c' \in C$ . If a chain possesses a maximal element then this is unique, and so will be referred to as the maximal element.

A partition  $\mathcal{C}$  of a subset  $F$  of  $E$  will be called a chain-partition of  $F$  if each element in  $\mathcal{C}$  is a chain. A subset  $D$  of  $E$  is called an antichain if no two distinct elements in  $D$  are comparable, i.e., if neither  $d \leq d'$  nor  $d' \leq d$  holds whenever  $d, d' \in D$  with  $d \neq d'$ . If  $D \subset F$  then we say that  $D$  is an antichain in  $F$ .

**Lemma 11.1** Let  $(E, \leq)$  be a poset and  $F \subset E$ . If  $\mathcal{C}$  is a chain-partition of  $F$  and  $D$  is an antichain in  $F$  then  $D \preceq \mathcal{C}$ .

In particular, if  $\mathcal{C}$  is a chain-partition of  $E$  and  $D$  is an antichain then  $D \preceq \mathcal{C}$ .

*Proof* Each chain  $C \in \mathcal{C}$  can contain at most one element of  $D$  and hence the restricted evaluation mapping  $i_C^D : D \rightarrow \mathcal{C}$  is injective.  $\square$

The following important result is Dilworth's decomposition theorem [2]. As stated at the beginning of the section, the proof presented here is due to Galvin [3].

**Theorem 11.1** *Let  $(A, \leq)$  be a finite poset. Then there exists a chain-partition  $\mathcal{C}$  of  $A$  and an antichain  $D$  such that  $D \approx \mathcal{C}$ .*

*Proof* We first need some preparation, and throughout the proof assume that  $(A, \leq)$  is a finite poset.

Let us say that a subset  $B$  of  $A$  is regular if there exists a chain-partition  $\mathcal{C}$  of  $B$  and an antichain  $D$  in  $B$  such that  $D \approx \mathcal{C}$ . We thus need to show that  $A$  itself is regular.

Let  $B$  be a regular subset of  $A$ . If  $\mathcal{C}$  and  $\mathcal{C}'$  are chain-partitions of  $B$  and  $D$  and  $D'$  are antichains in  $B$  with  $D \approx \mathcal{C}$  and  $D' \approx \mathcal{C}'$  then  $D \approx D' \approx \mathcal{C} \approx \mathcal{C}'$ , since by Lemma 11.1  $D \preceq \mathcal{C}' \approx D'$  and  $D' \preceq \mathcal{C} \approx D$  and hence by Theorem 2.4  $D \approx D'$ . We call  $\mathcal{C}$  and  $\mathcal{C}'$  minimal chain-partitions of  $B$  and  $D$  and  $D'$  maximal antichains in  $B$ . If  $\mathcal{C}$  is a minimal chain-partition of  $B$  then a chain-partition  $\mathcal{C}'$  of  $B$  is also minimal if and only if  $\mathcal{C}' \approx \mathcal{C}$ . In the same way, if  $D$  is a maximal antichain in  $B$  then an antichain  $D'$  in  $B$  is also maximal if and only if  $D' \approx D$ . If  $\mathcal{C}$  is any minimal chain-partition of  $B$  and  $D$  any maximal antichain in  $B$  then the restricted evaluation mapping  $i_C^D : D \rightarrow \mathcal{C}$  is a bijection.

**Lemma 11.2** *Let  $E$  be a subset of  $A$  such that every subset of  $E$  is regular. Then there exists a maximal antichain  $D_*$  in  $E$  and for each  $d \in D_*$  a minimal chain-partition  $\mathcal{C}_d$  of  $E$  such that the chain in  $\mathcal{C}_d$  containing  $d$  has  $d$  as its maximal element.*

*Proof* This holds trivially if  $E = \emptyset$  and so we can assume that  $E$  is non-empty. Denote the (non-empty) set of maximal antichains in  $E$  by  $\mathcal{D}$  and let  $\Delta$  be the union of all the sets in  $\mathcal{D}$ , i.e.,  $\Delta = \{e \in E : e \in D \text{ for some } D \in \mathcal{D}\}$ .

Now fix an arbitrary minimal chain-partition  $\mathcal{C}$  of  $E$ . For each  $C \in \mathcal{C}$  the set  $\Delta \cap C$  is non-empty (since it contains an element from each set in  $\mathcal{D}$ ), thus let  $m_C$  be the maximal element in  $\Delta \cap C$ . Finally let  $D_*$  be the set consisting of the elements  $m_C$ ,  $C \in \mathcal{C}$ .

We show that  $D_*$  is a maximal antichain in  $E$ : Let  $C_1, C_2 \in \mathcal{C}$  with  $C_1 \neq C_2$ . Then there exist  $D_1, D_2 \in \mathcal{D}$  such that  $m_{C_1}$  is the unique element in  $C_1 \cap D_1$  and

$m_{C_2}$  the unique element in  $C_2 \cap D_2$ . Let  $b_{12}$  be the unique element in  $C_1 \cap D_2$  and  $b_{21}$  the unique element in  $C_2 \cap D_1$ . Then  $b_{12} \leq m_{C_1}$ , since  $b_{12} \in \Delta \cap C_1$  and  $m_{C_1}$  is the maximal element of this set, and in the same way  $b_{21} \leq m_{C_2}$ . Now if  $m_{C_1} \leq m_{C_2}$  then it would follow that  $b_{12} \leq m_{C_2}$ , which is not possible since  $b_{12}$  and  $m_{C_2}$  are distinct elements of the antichain  $D_2$ . The same argument shows also that  $m_{C_2} \leq m_{C_1}$  is not possible, and hence  $D_*$  is an antichain. Clearly  $D_* \subset E$  and  $D_* \approx \mathcal{C}$ , since each chain in  $\mathcal{C}$  contains exactly one element of  $D_*$ . Therefore  $D_*$  is a maximal antichain in  $E$ .

Now for each  $d \in D_*$  we obtain a new minimal chain-partition  $\mathcal{C}_d$  of  $E$ . Let  $C_d$  be the chain in  $\mathcal{C}$  containing  $d$  and put  $C'_d = \{c \in C_d : c \leq d\}$ , thus  $C'_d$  is a non-empty chain. By assumption the set  $E_d = E \setminus C'_d$  is regular, and clearly  $D_* \setminus \{d\}$  is an antichain in  $E_d$ . Suppose there exists an antichain  $D'$  in  $E_d$  with  $D_* \setminus \{d\} \prec D'$ , i.e., with  $D_* \preceq D'$ . Then there is a subset  $D''$  of  $D'$  with  $D'' \approx D_*$  and  $D''$  is antichain in  $E_d$  and thus also an antichain in  $E$ , i.e.,  $D''$  is a maximal antichain in  $E$ . But this is not possible, since any maximal antichain in  $E$  intersects  $C'_d$ . It follows that  $D_* \setminus \{d\}$  is a maximal antichain in  $E_d$ . Let  $\mathcal{C}'_d$  be a minimal chain-partition of  $E_d$ ; thus  $\mathcal{C}'_d \approx D_* \setminus \{d\}$ . Finally  $\mathcal{C}_d = \mathcal{C}'_d \cup \{C'_d\}$  is a chain-partition of  $E$  and  $\mathcal{C}_d \approx \mathcal{C}'_d \cup \{C'_d\} \approx (D_* \setminus \{d\}) \cup \{d\} = D_*$ , i.e.,  $\mathcal{C}_d$  is a maximal chain-partition of  $E$ . Moreover,  $d$  is the maximal element of the chain  $C'_d$  in  $\mathcal{C}_d$ .  $\square$

Let  $\mathcal{S}$  be the set of regular subsets of  $A$ , and in particular  $\emptyset \in \mathcal{S}$ , since in this case the only chain-partition and antichain are empty. We will show that  $\mathcal{S} = \mathcal{P}(A)$  by applying Proposition 2.15. To do this we must show that each non-empty subset  $F$  of  $A$  contains an element  $s_F$  such that  $F \in \mathcal{S}$  whenever  $\mathcal{P}(F \setminus \{s_F\}) \subset \mathcal{S}$ , i.e., such that  $F$  is regular whenever each subset of  $F \setminus \{s_F\}$  is regular.

Thus let  $F$  be a non-empty subset of  $A$  and take  $s_F$  to be a maximal element of  $F$  (whose existence is guaranteed by Proposition 2.14). Suppose that each subset of  $E = F \setminus \{s_F\}$  is regular. Then by Lemma 11.2 there exists a maximal antichain  $D_*$  in  $E$  and for each  $d \in D_*$  a minimal chain-partition  $\mathcal{C}_d$  of  $E$  such that the chain in  $\mathcal{C}_d$  containing  $d$  has  $d$  as its maximal element.

Now if  $D_* \cup \{s_F\}$  is an antichain then it is immediate that  $F$  is regular, since  $\mathcal{C}_d \cup \{s_F\}$  is a chain-partition of  $F$  for any  $d \in D_*$  and  $\mathcal{C}_d \cup \{s_F\} \approx D_* \cup \{s_F\}$ . Thus suppose that  $D_* \cup \{s_F\}$  is not an antichain, and so there exists  $d \in D_*$  with  $d \leq s_F$  (since  $s_F$  is a maximal element of  $F$ ). Then  $\mathcal{C}_d \cup \{s_F\}$  is a chain, since  $d$  is the maximal element in  $C_d$  and  $d \leq s_F$ , which means  $(\mathcal{C}_d \setminus C_d) \cup (C_d \cup \{s_F\})$  is a chain-partition of  $F$ . But  $(\mathcal{C}_d \setminus C_d) \cup (C_d \cup \{s_F\}) \approx \mathcal{C}_d \approx D_*$  and  $D_*$  is also an antichain in  $F$ . It again follows that  $F$  is regular.

Proposition 2.15 now implies that  $\mathcal{S} = \mathcal{P}(A)$ . In particular  $A \in \mathcal{S}$  and so  $A$  is regular. The proof of Theorem 11.1 is complete.  $\square$

## 12 Enumerators

*In this section we give a further characterisation of a set being finite. This can be seen as having something to do with enumerating the elements in the set. Let us begin with a very informal discussion. Suppose we want to determine whether a given set  $E$  is finite or not. We could do this by marking the elements in  $E$  one at a time and seeing if all the elements can be marked in finitely many steps (whatever that means). At each stage of this process let us take a snapshot of the elements which have already been marked. This results in a subset  $\mathcal{U}$  of  $\mathcal{P}(E)$  whose elements are exactly these snapshots; each  $U \in \mathcal{U}$  is a subset of  $E$  specifying the elements of  $E$  which have already been marked at some stage in the process. The empty set  $\emptyset$  is in  $\mathcal{U}$  because we take a snapshot before marking the first element.*

*The following definition will be employed to help make the above more precise. First some notation: A subset  $\mathcal{U}$  of  $\mathcal{P}(E)$  is called an  $E$ -selector if  $\emptyset \in \mathcal{U}$  and for each  $U \in \mathcal{U}^p$  there exists a unique element  $e \in E \setminus U$  such that  $U \cup \{e\} \in \mathcal{U}$ . The set  $\mathcal{U}$  of snapshots should thus be an  $E$ -selector. Moreover, if  $E$  is finite then in the final snapshot all the elements of  $E$  will have been marked and so  $\mathcal{U}$  should contain  $E$ .*

*This suggests that a necessary condition for a set  $E$  to be finite is that there should exist an  $E$ -selector containing  $E$ . However, something is missing here since the condition as it stands is more-or-less vacuous since if  $\mathcal{U}$  is any  $E$ -selector then  $\mathcal{U} \cup \{E\}$  is also an  $E$ -selector which contains  $E$ . Thus the condition is just that there should exist an  $E$ -selector and it is clear that this does not ensure that the set  $E$  is finite since, for example,  $\{[n] : n \in \mathbb{N}\}$  is an  $\mathbb{N}$ -selector for the infinite set  $\mathbb{N}$ .*

*To see what is missing we need another definition. If  $\mathcal{U}$  is an  $E$ -selector then a subset  $\mathcal{V}$  of  $\mathcal{U}$  is said to be invariant if  $\emptyset \in \mathcal{V}$  and if  $V \cup \{e\} \in \mathcal{V}$  for all  $V \in \mathcal{V}^p$ , where  $e$  is the unique element of  $E \setminus V$  with  $V \cup \{e\} \in \mathcal{U}$ . In other words, a subset  $\mathcal{V}$  of  $\mathcal{U}$  is invariant if and only if it is itself an  $E$ -selector. An  $E$ -selector  $\mathcal{U}$  is said to be minimal if the only invariant subset of  $\mathcal{U}$  is  $\mathcal{U}$  itself. Note that any  $E$ -selector contains a unique subset which is a minimal  $E$ -selector and which can be obtained by taking the intersection of all its invariant subsets.*

*Now consider the  $E$ -selector  $\mathcal{U}$  described above whose elements are exactly the snapshots and let  $\mathcal{V}$  be any invariant subset of  $\mathcal{U}$ . Then the process of taking the snapshots produces elements of  $\mathcal{U}$  which start with the empty set  $\emptyset$ , which is in  $\mathcal{V}$ , and then, given that the current snapshot is an element of  $\mathcal{V}$ , will produce a new snapshot which is also an element of  $\mathcal{V}$ . (This follows from the definition of  $\mathcal{V}$  being invariant.) The process of taking snapshots can therefore only produce elements of  $\mathcal{U}$  which lie in  $\mathcal{V}$ . But if  $E$  is finite then this process produces all the*

elements of  $\mathcal{U}$ , and so in this case we must conclude that  $\mathcal{V} = \mathcal{U}$ . This means that  $\mathcal{U}$  must be a minimal  $E$ -selector. This suggests that a necessary condition for a set  $E$  to be finite is that there should exist a minimal  $E$ -selector containing  $E$  and such a minimal  $E$ -selector containing  $E$  will now be called an  $E$ -enumerator.

It turns out this necessary condition is also sufficient: In Theorem 12.1 it will be shown that an  $E$ -enumerator exists if and only if  $E$  is finite.

**Lemma 12.1** *If  $A$  is finite then every  $A$ -selector contains  $A$ .*

*Proof* An  $A$ -selector  $\mathcal{U}$  is a non-empty subset of  $\mathcal{P}(A)$  and thus by Proposition 1.3 it contains a maximal element  $U^*$ . But each  $U \in \mathcal{U}^p$  is not maximal, since there exists an element  $a \in A \setminus U$  with  $U \cup \{a\} \in \mathcal{U}$ . Hence  $U^* = A$ , and so  $\mathcal{U}$  contains  $A$ .  $\square$

Note that if  $A$  is finite then by Lemma 12.1 an  $A$ -selector is an  $A$ -enumerator if and only if it is minimal.

**Lemma 12.2** *If  $A$  is finite then there exists an  $A$ -selector.*

*Proof* Let  $\mathcal{S} = \{B \in \mathcal{P}(A) : \text{there exists a } B\text{-selector}\}$ . In particular  $\emptyset \in \mathcal{S}$ , since  $\{\emptyset\}$  is a  $\emptyset$ -selector. Let  $B \in \mathcal{S}^p$  and let  $a \in A \setminus B$ ; put  $B' = B \cup \{a\}$ . By assumption there exists a  $B$ -selector  $\mathcal{U}$  which by Lemma 12.1 contains  $B$ . It follows that  $\mathcal{U} \cup \{B'\}$  is a  $B'$ -selector and thus  $B' \in \mathcal{S}$ . This shows  $\mathcal{S}$  is an inductive  $A$ -system and hence  $\mathcal{S} = \mathcal{P}(A)$ , since  $A$  is finite. In particular  $A \in \mathcal{S}$  and so there exists an  $A$ -selector.  $\square$

**Theorem 12.1** *A set  $E$  is finite if and only if there exists an  $E$ -enumerator.*

*Proof* Suppose first that there exists an  $E$ -enumerator  $\mathcal{U}$  and let  $\mathcal{S}$  be an inductive  $E$ -system. Then  $\mathcal{U} \cap \mathcal{S}$  is an invariant subset of  $\mathcal{U}$  and therefore  $\mathcal{U} \cap \mathcal{S} = \mathcal{U}$ , since  $\mathcal{U}$  is minimal. Thus  $\mathcal{U} \subset \mathcal{S}$  and in particular  $E \in \mathcal{S}$ . This shows that each inductive  $E$ -system contains  $E$  and hence by Lemma 1.2  $E$  is finite. Suppose conversely that  $E$  is finite. By Lemma 12.2 there exists an  $E$ -selector and therefore there exists a minimal  $E$ -selector which by Lemma 1.2 contains  $E$ . This shows that an  $E$ -enumerator  $\mathcal{U}$  exists.  $\square$

A subset  $\mathcal{U}$  of  $\mathcal{P}(E)$  is said to be totally ordered if for all  $E_1, E_2 \in \mathcal{U}$  either  $E_1 \subset E_2$  or  $E_2 \subset E_1$ . Note that a set  $E$  is finite if and only if there exists a totally ordered  $E$ -enumerator. This follows from the fact that the  $E$ -selector obtained in Lemma 12.2 is totally ordered. In fact in Theorem 12.2 it will be shown that if  $A$  is finite then any  $A$ -enumerator is automatically totally ordered.

Note that  $\mathcal{U}^\emptyset = \{\emptyset\}$  is the single  $\emptyset$ -enumerator.



**Theorem 12.2** *If  $A$  is a finite set then an  $A$ -selector is minimal if and only if it is totally ordered and thus it is an  $A$ -enumerator if and only if it is totally ordered.*

*The proof requires some preparation. Throughout the section  $A$  always denotes a finite set.*

*For each  $E$ -selector  $\mathcal{U}$  let  $\mathbf{e}_{\mathcal{U}} : \mathcal{U}^p \rightarrow E$  and  $\mathbf{s}_{\mathcal{U}} : \mathcal{U}^p \rightarrow \mathcal{U} \setminus \{\emptyset\}$  be the mappings with  $\mathbf{e}_{\mathcal{U}}(U) = e$  and  $\mathbf{s}_{\mathcal{U}}(U) = U \cup \{e\}$ , where  $e$  is the unique element in  $E \setminus U$  such that  $U \cup \{e\} \in \mathcal{U}$ .*

**Lemma 12.3** *Let  $\mathcal{U}$  be a totally ordered  $A$ -selector and let  $U, U' \in \mathcal{U}$ . Then:*

- (1)  *$U'$  is a proper subset of  $U$  if and only if  $\mathbf{s}_{\mathcal{U}}(U') \subset U$ .*
- (2) *If  $U \in \mathcal{U}^p$  then  $U'$  is a subset of  $U$  if and only if it is a proper subset of  $\mathbf{s}_{\mathcal{U}}(U)$ .*

*Proof (1) If  $\mathbf{s}_{\mathcal{U}}(U') \subset U$  then  $U'$  is a proper subset of  $U$ , since  $\mathbf{e}_{\mathcal{U}}(U') \notin U'$ . Conversely, suppose  $U'$  is a proper subset of  $U$ . Then there is an injective mapping  $i : \mathbf{s}_{\mathcal{U}}(U') \rightarrow U$ . Thus if  $U \subset \mathbf{s}_{\mathcal{U}}(U')$  then by Proposition 2.4  $U = \mathbf{s}_{\mathcal{U}}(U')$  and in particular  $\mathbf{s}_{\mathcal{U}}(U') \subset U$ . But either  $U \subset \mathbf{s}_{\mathcal{U}}(U')$  or  $\mathbf{s}_{\mathcal{U}}(U') \subset U$ , and thus in both cases  $\mathbf{s}_{\mathcal{U}}(U') \subset U$ .*

*(2) If  $U' \subset U$  then  $U'$  is a proper subset of  $\mathbf{s}_{\mathcal{U}}(U)$ , since  $\mathbf{e}_{\mathcal{U}}(U) \notin U$ . Conversely, suppose  $U'$  is a proper subset of  $\mathbf{s}_{\mathcal{U}}(U)$ . Then there exists an injective mapping  $i : U' \rightarrow U$ . Thus if  $U \subset U'$  then  $U = U'$  and in particular  $U' \subset U$ . But either  $U \subset U'$  or  $U' \subset U$ , and thus in both cases  $U' \subset U$ .  $\square$*

**Lemma 12.4** *Let  $\mathcal{U}$  be a  $B$ -enumerator, where  $B$  is a non-empty finite set. Then  $u_0 = \mathbf{e}_{\mathcal{U}}(\emptyset) \in U$  for all  $U \in \mathcal{U} \setminus \{\emptyset\}$  and  $\mathcal{U}_0 = \{U \setminus \{u_0\} : U \in \mathcal{U} \setminus \{\emptyset\}\}$  is a  $(B \setminus \{u_0\})$ -enumerator. Moreover,  $\mathcal{U}$  is totally ordered if and only if  $\mathcal{U}_0$  is.*

*Proof The set  $\mathcal{V} = \{\emptyset\} \cup \{U \in \mathcal{U} : u_0 \in U\}$  contains  $\emptyset$  and if  $U \in \mathcal{V}^p$  then either  $U = \emptyset$ , in which case  $u_0 \in \{u_0\} = \mathbf{s}_{\mathcal{U}}(U)$ , or  $U \neq \emptyset$ , and then  $u_0 \in U \subset \mathbf{s}_{\mathcal{U}}(U)$ . Thus  $\mathcal{V}$  is an invariant subset of  $\mathcal{U}$  and so  $\mathcal{V} = \mathcal{U}$ , since  $\mathcal{U}$  is minimal. Hence  $u_0 \in U$  for all  $U \in \mathcal{U} \setminus \{\emptyset\}$ . Now  $\mathbf{e}_{\mathcal{U}}(U) \neq u_0$  whenever  $U \in \mathcal{U} \setminus \{\emptyset\}$  (since  $\mathbf{e}_{\mathcal{U}}(U) \in B \setminus U$  and  $u_0 \in U$ ), and so  $\mathcal{U}_0$  is a  $(B \setminus \{u_0\})$ -selector. Moreover, if  $\mathcal{V}_0$  is an invariant subset of  $\mathcal{U}_0$  then  $\mathcal{V}'_0 = \{\emptyset\} \cup \{U \cup \{u_0\} : U \in \mathcal{V}_0\}$  is an invariant subset of  $\mathcal{U}$ . Therefore  $\mathcal{V}'_0 = \mathcal{U}$ , since  $\mathcal{U}$  is minimal, which implies that  $\mathcal{V}_0 = \mathcal{U}_0$ . This shows that  $\mathcal{U}_0$  is a  $(B \setminus \{u_0\})$ -enumerator. Finally, it is clear that  $\mathcal{U}$  is totally ordered if and only if  $\mathcal{U}_0$  is.  $\square$*

*Proof of Theorem 12.2* Assume first there exists an  $A$ -enumerator which is not totally ordered. Then the subset  $\mathcal{S}$  of  $\mathcal{P}(A)$  consisting of those subsets  $B$  for which there exists a  $B$ -enumerator which is not totally ordered is non-empty. Hence by Proposition 1.2  $\mathcal{S}$  contains a minimal element  $B$  and  $B$  is non-empty since the only  $\emptyset$ -enumerator is trivially totally ordered. There thus exists a  $B$ -enumerator  $\mathcal{U}$  which is not totally ordered and then Lemma 12.4 implies that  $B \setminus \{u_0\} \in \mathcal{S}$ , where  $u_0 = e_{\mathcal{U}}(\emptyset)$ . This contradicts the minimality of  $B$  and therefore the assumption that there exists an  $A$ -enumerator which is not totally ordered is false. In other words, each  $A$ -enumerator is totally ordered.

For the converse let  $\mathcal{U}$  be a totally ordered  $A$ -selector and suppose there exists an invariant proper subset  $\mathcal{V}$  of  $\mathcal{U}$ . Put  $\mathcal{S} = \mathcal{U} \setminus \mathcal{V}$ . Then  $\mathcal{S}$  is non-empty and hence by Proposition 1.2 it contains a minimal element  $U_*$ , and  $U_* \neq \emptyset$ , since  $\emptyset \notin \mathcal{S}$ . Put  $\mathcal{R} = \{V \in \mathcal{V} : V \subset U_*\}$ ; then  $\mathcal{R}$  is non-empty (since it contains  $\emptyset$ ) and therefore by Proposition 1.3  $\mathcal{R}$  contains a maximal element  $U^*$ . Thus  $U^* \subset U_*$ , and in fact  $U^*$  is a proper subset of  $U_*$ , since  $U^* \in \mathcal{V}$  and  $U_* \notin \mathcal{V}$ . Hence by Lemma 12.3 (1)  $s_{\mathcal{U}}(U^*) \subset U_*$ . But  $s_{\mathcal{U}}(U^*) \in \mathcal{V}$ , since  $\mathcal{V}$  is invariant, and so  $s_{\mathcal{U}}(U^*) \in \mathcal{R}$ . However, this contradicts the maximality of  $U^*$  and we conclude that  $\mathcal{V} = \mathcal{U}$ . Therefore  $\mathcal{U}$  is minimal, i.e.,  $\mathcal{U}$  is an  $A$ -enumerator.

This completes the proof of Theorem 12.2.  $\square$

If  $\mathcal{U}$  is an  $A$ -enumerator then for each  $U \in \mathcal{U}$  the set  $\mathcal{U} \cap \mathcal{P}(U)$  will be denoted by  $\mathcal{U}_U$ . Note that, as far as the definition of  $\mathcal{U}_U^p$  is concerned,  $\mathcal{U}_U$  is considered here to be a subset of  $\mathcal{P}(U)$  and so  $\mathcal{U}_U^p = \{U' \in \mathcal{U} : U' \text{ is a proper subset of } U\}$ . If  $U' \in \mathcal{U}_U^p$  then by Lemma 12.3 (1)  $s_{\mathcal{U}}(U') \in \mathcal{U}_U$  and so  $\mathcal{U}_U$  is a  $U$ -selector. But  $\mathcal{U}_U$  is clearly totally ordered and therefore by Theorem 12.2 it is in fact a  $U$ -enumerator. If  $U \in \mathcal{U}^p$  and  $U^* = s_{\mathcal{U}}(U)$  then by Lemma 12.3 (2)  $\mathcal{U}_{U^*}^p = \mathcal{U}_U$ .

**Lemma 12.5** *If  $\mathcal{U}$  is an  $A$ -enumerator then for all  $U \in \mathcal{U}$*

$$U = \{a \in A : a = e_{\mathcal{U}}(U') \text{ for some } U' \in \mathcal{U}_U^p\}.$$

*Proof* Let  $\mathcal{V}$  be the set consisting of those elements  $U \in \mathcal{U}$  for which the statement above holds, i.e., for which  $U = \{a \in A : a = e_{\mathcal{U}}(U') \text{ for some } U' \in \mathcal{U}_U^p\}$ , and hence in particular  $\emptyset \in \mathcal{V}$ . Let  $U \in \mathcal{V}^p$  and put  $U_* = s_{\mathcal{U}}(U)$ ; then by Lemma 12.3 (2)

$$\begin{aligned} s_{\mathcal{U}}(U) &= U \cup \{e_{\mathcal{U}}(U)\} = \{a \in A : a = e_{\mathcal{U}}(U') \text{ for some } U' \in \mathcal{U}_U\} \\ &= \{a \in A : a = e_{\mathcal{U}}(U') \text{ for some } U' \in \mathcal{U}_{U_*}^p\} \end{aligned}$$

and hence  $s_{\mathcal{U}}(U) = U_* \in \mathcal{V}$ . Thus  $\mathcal{V}$  is an invariant subset of  $\mathcal{U}$ , and so  $\mathcal{V} = \mathcal{U}$ .

$\square$

**Proposition 12.1** *If  $\mathcal{U}$  is an  $A$ -enumerator then the mappings  $s_{\mathcal{U}} : \mathcal{U}^p \rightarrow \mathcal{U} \setminus \{\emptyset\}$  and  $e_{\mathcal{U}} : \mathcal{U}^p \rightarrow A$  are both bijections. In particular, if  $\mathcal{U}$  and  $\mathcal{V}$  are  $A$ -enumerators then  $\mathcal{U} \approx \mathcal{V}$ . (This means, somewhat imprecisely, that any  $A$ -enumerator contains one more element than  $A$ .)*

*Proof* Let  $\mathcal{V} = \{\emptyset\} \cup \{U \in \mathcal{U} : \text{there exists } U' \in \mathcal{U}^p \text{ such that } U = s_{\mathcal{U}}(U')\}$ . Then  $\emptyset \in \mathcal{V}$  and  $s_{\mathcal{U}}(U)$  is trivially an element of  $\mathcal{V}$  for all  $U \in \mathcal{U}^p$ , and so in particular for all  $U \in \mathcal{V}^p$ . Therefore  $\mathcal{V}$  is an invariant subset of  $\mathcal{U}$  and so  $\mathcal{V} = \mathcal{U}$ . This shows that the mapping  $s_{\mathcal{U}}$  is surjective. Now consider the mapping  $s'_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{U}$  with  $s'_{\mathcal{U}}(A) = \emptyset$  and  $s'_{\mathcal{U}}(U) = s_{\mathcal{U}}(U)$  whenever  $U \in \mathcal{U}^p$ . Then  $s'_{\mathcal{U}}$  is surjective, since  $s_{\mathcal{U}}$  is, and hence by Theorem 2.1  $s'_{\mathcal{U}}$  is bijective, since  $\mathcal{U}$  is finite. It follows that  $s_{\mathcal{U}}$  is also bijective.

Now to the mapping  $e_{\mathcal{U}}$ . Let  $U, U' \in \mathcal{U}^p$  with  $U \neq U'$ . By Theorem 12.2  $\mathcal{U}$  is totally ordered, thus either  $U \subset U'$  or  $U' \subset U$  and so without loss of generality assume that  $U' \subset U$ . Therefore  $U'$  is a proper subset of  $U$ , hence by Lemma 12.3 (1)  $s_{\mathcal{U}}(U') \subset U$  and in particular  $e_{\mathcal{U}}(U') \in U$ . But  $e_{\mathcal{U}}(U) \notin U$ , which implies that  $e_{\mathcal{U}}(U) \neq e_{\mathcal{U}}(U')$ . This shows the mapping  $e_{\mathcal{U}}$  is injective. Moreover, by Lemma 12.5 (with  $U = A$ )  $A = \{a \in A : a = e_{\mathcal{U}}(U) \text{ for some } U \in \mathcal{U}^p\}$ , and thus the mapping  $e_{\mathcal{U}}$  is surjective.  $\square$

**Lemma 12.6** *If  $B \subset A$  then there exists an  $A$ -enumerator  $\mathcal{U}$  with  $B \in \mathcal{U}$ .*

*Proof* Let  $\mathcal{V}$  be a  $B$ -enumerator and  $\mathcal{V}'$  be an  $(A \setminus B)$ -enumerator. Then by Theorem 12.2  $\mathcal{U} = \mathcal{V} \cup \{B \cup C : C \in \mathcal{V}' \setminus \{\emptyset\}\}$  is clearly a totally ordered  $A$ -selector containing  $B$  and thus by Theorem 12.2 it is an  $A$ -enumerator containing  $B$ .  $\square$

In what follows  $B$  is always a finite set.

If  $\mathcal{U}$  is an  $A$ -enumerator and  $\mathcal{V}$  a  $B$ -enumerator then a mapping  $\pi : \mathcal{U} \rightarrow \mathcal{V}$  is called a homomorphism if  $\pi(\emptyset) = \emptyset$ ,  $\pi(\mathcal{U}^p) \subset \mathcal{V}^p$  and  $\pi(s_{\mathcal{U}}(U)) = s_{\mathcal{V}}(\pi(U))$  for all  $U \in \mathcal{U}^p$ .

**Proposition 12.2** *If  $\pi : \mathcal{U} \rightarrow \mathcal{V}$  is a homomorphism from an  $A$ -enumerator  $\mathcal{U}$  to a  $B$ -enumerator  $\mathcal{V}$  then  $\pi(U) \approx U$  for all  $U \in \mathcal{U}$  and  $\pi$  maps  $\mathcal{U}$  bijectively onto  $\mathcal{V}_{\pi(A)}$ .*

*Proof* Let  $\mathcal{U}_0$  denote the set consisting of those  $U \in \mathcal{U}$  for which  $\pi(U') \approx U'$  for all  $U' \in \mathcal{U}_U$  and for which  $\pi$  maps  $\mathcal{U}_U$  bijectively onto  $\mathcal{V}_{\pi(U)}$ . Clearly  $\emptyset \in \mathcal{U}_0$  since  $\pi(\emptyset) = \emptyset$  and  $\mathcal{U}_{\emptyset} = \mathcal{V}_{\pi(\emptyset)} = \{\emptyset\}$ .

Consider  $U \in \mathcal{U}_0^p$  and put  $U^* = s_{\mathcal{U}}(U)$ ; by Lemma 12.3 (2)  $\mathcal{U}_{U^*} = \mathcal{U}_U \cup \{U^*\}$ . Now  $\pi(U^*) = \pi(s_{\mathcal{U}}(U)) = s_{\mathcal{V}}(\pi(U)) = \pi(U) \cup \{c\}$ , where  $c \notin \pi(U)$ ,  $U^* = U \cup \{d\}$ , where  $d \notin U$  and  $\pi(U) \approx U$ , since  $U \in \mathcal{U}_U$ . It follows that  $\pi(U^*) \approx U^*$ , and hence  $\pi(U') \approx U'$  for all  $U' \in \mathcal{U}_{U^*}$ . Moreover,  $\mathcal{V}_{\pi(U^*)} = \mathcal{V}_{s_{\mathcal{V}}(\pi(U))} = \mathcal{V}_{\pi(U)} \cup \{s_{\mathcal{V}}(\pi(U))\}$ ,  $\pi(U^*) = s_{\mathcal{V}}(\pi(U))$  and  $\pi$  maps  $\mathcal{U}_U$  bijectively onto  $\mathcal{V}_{\pi(U)}$ . It follows that  $\pi$  maps  $\mathcal{U}_{U^*}$  bijectively onto  $\mathcal{V}_{\pi(U^*)}$ . This shows  $U^* = s_{\mathcal{U}}(U) \in \mathcal{U}_0$ , and so  $\mathcal{U}_0$  is an invariant subset of  $\mathcal{U}$ . Thus  $\mathcal{U}_0 = \mathcal{U}$  and then by Lemma 12.1  $A \in \mathcal{U}_0$ , i.e.,  $\pi(U) \approx U$  for all  $U \in \mathcal{U}$  and  $\pi$  maps  $\mathcal{U}$  bijectively onto  $\mathcal{V}_{\pi(A)}$ .  $\square$

If  $\pi : \mathcal{U} \rightarrow \mathcal{V}$  is a homomorphism as above then by Proposition 12.1  $\pi(A) \approx A$ . But  $\pi(A)$  is a subset of  $B$ , hence  $\pi(A) \preceq B$  and thus  $A \preceq B$ . This necessary condition for the existence of a homomorphism is also sufficient:

**Proposition 12.3** *If  $A \preceq B$ ,  $\mathcal{U}$  is an  $A$ -enumerator and  $\mathcal{V}$  a  $B$ -enumerator then there exists a unique homomorphism  $\pi : \mathcal{U} \rightarrow \mathcal{V}$ .*

*Proof* Let  $\mathcal{U}_0$  denote the set consisting of those  $U \in \mathcal{U}$  for which there exists a homomorphism  $\pi_U : \mathcal{U}_U \rightarrow \mathcal{V}$ . Clearly  $\emptyset \in \mathcal{U}_0$  since  $\mathcal{U}_{\emptyset} = \{\emptyset\}$  and  $\mathcal{U}_{\emptyset}^p = \emptyset$ .

Consider  $U \in \mathcal{U}_0^p$  and let  $\pi_U : \mathcal{U}_U \rightarrow \mathcal{V}$  be a homomorphism. Now  $A \preceq B$  and  $U$  is a proper subset of  $A$  and hence  $U \not\approx B$ ; it follows that  $\pi_U(U) \neq B$ , since by Proposition 12.1  $\pi_U(U) \approx U$ . Let  $U^* = s_{\mathcal{U}}(U)$ ; by Lemma 12.3 (2)  $\mathcal{U}_{U^*} = \mathcal{U}_U \cup \{U^*\}$  and so we can define  $\pi_{U^*} : \mathcal{U}_{U^*} \rightarrow \mathcal{V}$  by putting  $\pi_{U^*}(U') = \pi_U(U')$  if  $U' \in \mathcal{U}_U$  and letting  $\pi_{U^*}(U^*) = s_{\mathcal{V}}(\pi_U(U))$  (recalling that  $\pi_U(U) \neq B$ ). If  $U' \in \mathcal{U}_{U^*}^p$  then  $U' \in \mathcal{U}_U$  and  $s_{\mathcal{U}}(U') \in \mathcal{U}_U$  and thus

$$\pi_{U^*}(s_{\mathcal{U}}(U')) = \pi_U(s_{\mathcal{U}}(U')) = s_{\mathcal{V}}(\pi_U(U')) = s_{\mathcal{V}}(\pi_{U^*}(U')) .$$

Also  $\pi_{U^*}(s_{\mathcal{U}}(U)) = \pi_{U^*}(U^*) = s_{\mathcal{V}}(\pi_U(U)) = s_{\mathcal{V}}(\pi_{U^*}(U))$  and  $\mathcal{U}_U = \mathcal{U}_{U^*}^p$ , and thus  $\pi_{U^*}(s_{\mathcal{U}}(U')) = s_{\mathcal{V}}(\pi_{U^*}(U'))$  for all  $U' \in \mathcal{U}_{U^*}^p$ . Hence  $\pi_{U^*}$  is a homomorphism and so  $s_{\mathcal{U}}(U) = U^* \in \mathcal{U}_0$ . This shows  $\mathcal{U}_0$  is an invariant subset of  $\mathcal{U}$ . It follows that  $\mathcal{U}_0 = \mathcal{U}$  and then by Lemma 12.1  $B \in \mathcal{U}_0$ , which means that  $\pi = \pi_B : \mathcal{U} \rightarrow \mathcal{V}$  is a homomorphism.

It remains to consider the uniqueness. Let  $\pi' : \mathcal{U} \rightarrow \mathcal{V}$  be any homomorphism and put  $\mathcal{U}_0 = \{U \in \mathcal{U} : \pi'(U) = \pi(U)\}$ . Clearly  $\emptyset \in \mathcal{U}_0$  and if  $U \in \mathcal{U}_0^p$  then  $\pi'(s_{\mathcal{U}}(U)) = s_{\mathcal{V}}(\pi'(U)) = s_{\mathcal{V}}(\pi(U)) = \pi(s_{\mathcal{U}}(U))$ , i.e.,  $s_{\mathcal{U}}(U) \in \mathcal{U}_0$ . Thus  $\mathcal{U}_0$  is an invariant subset of  $\mathcal{U}$ , and so  $\mathcal{U}_0 = \mathcal{U}$ . This shows that  $\pi' = \pi$ , i.e., there is a unique homomorphism  $\pi : \mathcal{U} \rightarrow \mathcal{V}$ .  $\square$

**Theorem 12.3** *If  $A \approx B$ ,  $\mathcal{U}$  is an  $A$ -enumerator and  $\mathcal{V}$  is a  $B$ -enumerator then there exists a unique homomorphism  $\pi : \mathcal{U} \rightarrow \mathcal{V}$  and  $\pi$  maps  $\mathcal{U}$  bijectively onto  $\mathcal{V}$ . Moreover,  $\pi(A) = B$ .*

*Proof* This follows from Propositions 12.2 and 12.3. (Note that  $\pi(A)$  is a subset of  $B$  with  $\pi(A) \approx A$  and  $A \approx B$  and thus with  $\pi(A) \approx B$ . Hence by Theorem 2.2  $\pi(A) = B$ .)  $\square$

An important special case of Theorem 12.3 is when there is a second  $A$ -enumerator  $\mathcal{U}'$ . There then exists a unique homomorphism  $\pi : \mathcal{U} \rightarrow \mathcal{U}'$ ,  $\pi$  maps  $\mathcal{U}$  bijectively onto  $\mathcal{U}'$  and  $\pi(A) = A$ .

Let  $\mathcal{U}$  be an  $A$ -enumerator. By Proposition 12.1 the mapping  $e_{\mathcal{U}} : \mathcal{U}^p \rightarrow A$  is a bijection and so there is a unique binary relation  $\leq$  on  $A$  such that  $e_{\mathcal{U}}(U) \leq e_{\mathcal{U}}(U')$  holds for  $U, U' \in \mathcal{U}^p$  if and only if  $U \subset U'$ . More explicitly, this means that  $a \leq a'$  if and only if  $e_{\mathcal{U}}^{-1}(a) \subset e_{\mathcal{U}}^{-1}(a')$ , where  $e_{\mathcal{U}}^{-1} : A \rightarrow \mathcal{U}^p$  is the inverse of the mapping  $e_{\mathcal{U}}$ . It is clear that  $\leq$  is a total order and it will be called the total order associated with  $\mathcal{U}$ .

**Proposition 12.4** Let  $\leq$  be a total order on  $A$  and put  $L_a = \{a' \in A : a' < a\}$  for each  $a \in A$  (where as usual  $a' < a$  means that both  $a' \leq a$  and  $a' \neq a$  hold). Then  $\mathcal{U} = \{U \in \mathcal{P}(A) : U = L_a \text{ for some } a \in A\} \cup \{A\}$  is an  $A$ -enumerator with  $e_{\mathcal{U}}(L_a) = a$  for each  $a \in A$ .

*Proof* We assume that  $A$  is non-empty, since the result holds trivially when  $A = \emptyset$ . By Proposition 2.14 the non-empty set  $A$  contains a unique  $\leq$ -minimum element  $a_0$  and then  $L_{a_0} = \emptyset$ , which shows that  $\emptyset \in \mathcal{U}$ . Let  $U \in \mathcal{U}^p$  and let  $a \in A$  be such that  $U = L_a$ ; put  $U' = U \cup \{a\}$ . If  $U' = A$  then  $a$  is trivially the unique element in  $A \setminus U$  with  $U \cup \{a\} \in \mathcal{U}$ , so consider the case with  $U' \neq A$ . Then by Proposition 2.14 the non-empty set  $A \setminus U'$  contains a unique  $\leq$ -minimum element  $a'$ . Now  $U' = \{b \in A : b \leq a\}$ , thus  $A \setminus U' = \{b \in A : a < b\}$  and so  $U' \subset L_{a'}$ . But  $a < c < a'$  for each  $c \in L_{a'} \setminus U'$  and hence  $L_{a'} \setminus U' = \emptyset$ , since  $a'$  is the  $\leq$ -minimum element in  $\{b \in A : a < b\}$ . It follows that  $U' = L_{a'}$ , i.e.,  $U \cup \{a\} \in \mathcal{U}$ . Suppose  $U \cup \{b\} \in \mathcal{U}$  for some other  $b \in A \setminus U$ . Then  $U \cup \{b\} = L_{b'}$  for some  $b' \in A$  (since  $U \cup \{b\} \neq A$ ) and then  $a \leq b < a'$ , since  $L_a \subset L_{a'}$  and  $b \in L_{a'} \setminus L_a$ . But this implies  $a \in L_{a'}$  and hence  $b = a$ , since  $a \notin L_a$ . Thus  $a$  is the unique element in  $A \setminus U$  such that  $U \cup \{a\} \in \mathcal{U}$ . This shows that  $\mathcal{U}$  is an  $A$ -selector (since it is clearly totally ordered) and that  $e_{\mathcal{U}}(L_a) = a$  for each  $a \in A$ . Moreover,  $\mathcal{U}$  Hence by Theorem 12.2  $\mathcal{U}$  is an  $A$ -enumerator.  $\square$

If  $\leq$  is a total order on  $A$  then the  $A$ -enumerator  $\mathcal{U}$  given in Proposition 12.4 will be called the  $A$ -enumerator associated with  $\leq$ .

**Theorem 12.4** (1) If  $\leq$  is the total order associated with an  $A$ -enumerator  $\mathcal{U}$  then  $\mathcal{U}$  is the  $A$ -enumerator associated with  $\leq$ .

(2) If  $\mathcal{U}$  is the  $A$ -enumerator associated with a total order  $\leq$  on  $A$  then  $\leq$  is the total order associated with  $\mathcal{U}$ .

*Proof (1)* Let  $\leq$  be the total order associated with the  $A$ -enumerator  $\mathcal{U}$  and let  $\mathcal{V}$  be the  $A$ -enumerator associated with  $\leq$ . Thus if  $U, U' \in \mathcal{U}^p$  then  $\mathbf{e}_{\mathcal{U}}(U) \leq \mathbf{e}_{\mathcal{U}}(U')$  if and only if  $U \subset U'$ . Now  $\mathcal{V} = \{V \in \mathcal{P}(A) : U = L_a \text{ for some } a \in A\} \cup \{A\}$ , where  $L_a = \{a' \in A : a' < a\}$  for each  $a \in A$ . Let  $U \in \mathcal{U}^p$  and put  $a = \mathbf{e}_{\mathcal{U}}(U)$ . Then

$$L_a = \{a' \in A : a' < a\} = \{b \in A : b = \mathbf{e}_{\mathcal{U}}(U') \text{ for some } U' \in \mathcal{U}_U^p\}$$

and therefore by Lemma 12.5  $L_a = U$ . It follows that  $\mathcal{V} = \mathcal{U}$ .

(2) Let  $\mathcal{U}$  be the  $A$ -enumerator associated with the total order  $\leq$  and let  $\leq'$  be the total order associated with  $\mathcal{U}$ . Thus if  $U, U' \in \mathcal{U}^p$  then there exist  $a, a' \in A$  with  $U = L_a$  and  $U' = L_{a'}$  and by Proposition 12.4  $\mathbf{e}_{\mathcal{U}}(U) = a$  and  $\mathbf{e}_{\mathcal{U}}(U') = a'$ . Thus  $a \leq' a'$  if and only if  $U \subset U'$  which means that  $a \leq' a'$  if and only if  $\{b \in A : b < a\} \subset \{b \in A : b < a'\}$ . Suppose  $a \leq a'$ . If  $b < a$  then  $b < a'$  and so  $\{b \in A : b < a\} \subset \{b \in A : b < a'\}$ . Conversely, suppose  $\{b \in A : b < a\} \subset \{b \in A : b < a'\}$ . Now either  $a \leq a'$  or  $a' \leq a$  and if  $a' \leq a$  then  $\{b \in A : b < a'\} \subset \{b \in A : b < a\}$  from which it follows that  $\{b \in A : b < a\} = \{b \in A : b < a'\}$  and this is only possible if  $a = a'$ . Therefore  $\leq' = \leq$ .  $\square$

We end the section by looking at the relationship between enumerators and iterators and in what follows let  $\mathbb{I} = (X, f, x_0)$  be a fixed iterator, where for simplicity we assume that the first component  $X$  is a set. Let  $\omega$  be the assignment of finite sets in  $\mathbb{I}$ . For the finite set  $A$  the element  $\omega(A)$  can be thought of as the analogue of the cardinality of  $A$  for the iterator  $\mathbb{I}$ . Now the cardinality  $|A|$  of a finite set  $A$  can also be determined by counting or enumerating its elements and the analogue of this procedure can also be carried out in the iterator  $\mathbb{I}$ . To explain what this means let us return to the informal discussion presented at the beginning of the section. There we determined whether  $A$  is finite or not by marking the elements in  $A$  one at a time and seeing if all the elements can be marked in finitely many steps. At each stage of this process we took a snapshot of the elements which have already been marked, which resulted in the  $A$ -enumerator  $\mathcal{U}$  whose elements are exactly the snapshots. The set  $A$  being finite meant that  $A \in \mathcal{U}$ .

Now suppose that each act of marking an element of  $A$  is registered with the iterator  $\mathbb{I}$ . Each such act produces an element of  $X$  which can be considered as the current state of the registering process. Before the first element of  $A$  has been marked the current state is  $x_0$ . If at some stage the current state is  $x$  then marking the next element of  $A$  changes the current state to  $f(x)$ .

This registering process can be regarded as a mapping  $\alpha_{\mathcal{U}} : \mathcal{U} \rightarrow X$ , where  $\alpha_{\mathcal{U}}(U)$  gives the current state when the elements in the subset  $U$  have been marked. The

above interpretation then requires that  $\alpha_{\mathcal{U}}(\emptyset) = x_0$  and  $\alpha_{\mathcal{U}}(s_{\mathcal{U}}(U)) = f(\alpha_{\mathcal{U}}(U))$  for all  $U \in \mathcal{U}^p$ . In Proposition 12.5 it is shown that there is a unique mapping  $\alpha_{\mathcal{U}}$  satisfying these requirements.

Since  $A$  is finite the registering process ends when all the elements in  $A$  have been marked and the final current state is then the element  $\alpha_{\mathcal{U}}(A)$  of  $X$ . Now if the analogy with the iterator  $(\mathbb{N}, s, 0)$  and the cardinality  $|A|$  is valid then we would expect that  $\alpha_{\mathcal{U}}(A) = \omega(A)$  holds for each finite set. In fact this does hold, as is shown in Theorem 12.5.

Theorem 12.5 implies that  $\alpha_{\mathcal{U}}(A)$  does not depend on the  $A$ -enumerator  $\mathcal{U}$ . This is the fundamental reason why counting makes sense: It does not matter in which order the elements in a finite set are counted; the same number always comes out in the end.

**Proposition 12.5** *Let  $\mathcal{U}$  be an  $A$ -enumerator. Then there exists a unique mapping  $\alpha_{\mathcal{U}} : \mathcal{U} \rightarrow X$  with  $\alpha_{\mathcal{U}}(\emptyset) = x_0$  such that  $\alpha_{\mathcal{U}}(s_{\mathcal{U}}(U)) = f(\alpha_{\mathcal{U}}(U))$  for all  $U \in \mathcal{U}^p$ .*

*Proof* This is essentially the same as the proof of Theorem 12.3. Let  $\mathcal{U}_0$  denote the set consisting of those  $U \in \mathcal{U}$  for which there exists a mapping  $\alpha_U : \mathcal{U}_U \rightarrow X$  with  $\alpha_U(\emptyset) = x_0$  and such that  $\alpha_U(s_{\mathcal{U}}(U')) = f(\alpha_U(U'))$  for all  $U' \in \mathcal{U}_U^p$ . Clearly  $\emptyset \in \mathcal{U}_0$  (since  $\mathcal{U}_{\emptyset} = \{\emptyset\}$ ).

Consider  $U \in \mathcal{U}_0^p$  and let  $\alpha_U : \mathcal{U}_U \rightarrow X$  be a mapping with  $\alpha_U(\emptyset) = x_0$  and such that  $\alpha_U(s_{\mathcal{U}}(U')) = f(\alpha_U(U'))$  for all  $U' \in \mathcal{U}_U^p$ . Write  $U^*$  for  $s_{\mathcal{U}}(U)$ ; by Lemma 12.3 (2)  $\mathcal{U}_{U^*} = \mathcal{U}_U \cup \{U^*\}$  and so we can define  $\alpha_{U^*} : \mathcal{U}_{U^*} \rightarrow X$  by putting  $\alpha_{U^*}(U') = \alpha_U(U')$  if  $U' \in \mathcal{U}_U$  and letting  $\alpha_{U^*}(U^*) = f(\alpha_U(U))$ . If  $U' \in \mathcal{U}_U^p$  then  $U' \in \mathcal{U}_{U^*}$  and  $s_{\mathcal{U}}(U') \in \mathcal{U}_U$  and thus

$$\alpha_{U^*}(s_{\mathcal{U}}(U')) = \alpha_U(s_{\mathcal{U}}(U')) = f(s_{\mathcal{U}}(U')) = f(\alpha_{U^*}(U')) .$$

Also  $\alpha_{U^*}(s_{\mathcal{U}}(U)) = \alpha_{U^*}(U^*) = f(\alpha_U(U)) = f(\alpha_{U^*}(U))$  and  $\mathcal{U}_U = \mathcal{U}_{U^*}^p$ , and so  $\alpha_{U^*}(s_{\mathcal{U}}(U')) = f(\alpha_{U^*}(U'))$  for all  $U' \in \mathcal{U}_{U^*}^p$ . Hence  $s_{\mathcal{U}}(U) = U^* \in \mathcal{U}_0$ . This shows  $\mathcal{U}_0$  is an invariant subset of  $\mathcal{U}$ . Thus  $\mathcal{U}_0 = \mathcal{U}$  and then by Lemma 12.1  $A \in \mathcal{U}_0$ , which means there exists a mapping  $\alpha_{\mathcal{U}} : \mathcal{U} \rightarrow X$  with  $\alpha_{\mathcal{U}}(\emptyset) = x_0$  and such that  $\alpha_{\mathcal{U}}(s_{\mathcal{U}}(U)) = f(\alpha_{\mathcal{U}}(U))$  for all  $U \in \mathcal{U}^p$ .

It remains to consider the uniqueness. Let  $\alpha'_{\mathcal{U}} : \mathcal{U} \rightarrow X$  be any mapping with  $\alpha'_{\mathcal{U}}(\emptyset) = x_0$  and such that  $\alpha'_{\mathcal{U}}(s_{\mathcal{U}}(U)) = f(\alpha'_{\mathcal{U}}(U))$  for all  $U \in \mathcal{U}^p$  and consider the set  $\mathcal{U}_0 = \{U \in \mathcal{U} : \alpha'_{\mathcal{U}}(U) = \alpha_{\mathcal{U}}(U)\}$ . Clearly  $\emptyset \in \mathcal{U}_0$  and if  $U \in \mathcal{U}_0^p$  then  $\alpha'_{\mathcal{U}}(s_{\mathcal{U}}(U)) = f(\alpha'_{\mathcal{U}}(U)) = f(\alpha_{\mathcal{U}}(U)) = \alpha_{\mathcal{U}}(s_{\mathcal{U}}(U))$ , i.e.,  $s_{\mathcal{U}}(U) \in \mathcal{U}_0$ . Thus  $\mathcal{U}_0$  is an invariant subset of  $\mathcal{U}$ , and so  $\mathcal{U}_0 = \mathcal{U}$ . This shows that  $\alpha'_{\mathcal{U}} = \alpha_{\mathcal{U}}$ .  $\square$

The mapping  $\alpha_{\mathcal{U}} : \mathcal{U} \rightarrow X$  in Proposition 12.5 will be referred to as the  $\mathcal{U}$ -valuation in  $\mathbb{I}$ , or just as the  $\mathcal{U}$ -valuation if it is clear which iterator is involved.

The uniqueness of the  $\mathcal{U}$ -valuation implies that for each  $U \in \mathcal{U}$  the  $\mathcal{U}_U$ -valuation  $\alpha_{\mathcal{U}_U}$  is just the restriction of  $\alpha_{\mathcal{U}}$  to  $\mathcal{U}_U$ , i.e.,  $\alpha_{\mathcal{U}_U} : \mathcal{U}_U \rightarrow X$  is the mapping with  $\alpha_{\mathcal{U}_U}(U') = \alpha_{\mathcal{U}}(U')$  for all  $U' \in \mathcal{U}_U$ .

**Proposition 12.6** Let  $\mathcal{U}$  be an  $A$ -enumerator and  $\mathcal{V}$  a  $B$ -enumerator. Suppose  $A \preceq B$  and so by Proposition 12.3 there exists a unique homomorphism  $\pi : \mathcal{U} \rightarrow \mathcal{V}$ . Then  $\alpha_{\mathcal{U}} = \alpha_{\mathcal{V}} \circ \pi$  (with  $\alpha_{\mathcal{U}}$  the  $\mathcal{U}$ -valuation and  $\alpha_{\mathcal{V}}$  the  $\mathcal{V}$ -valuation).

*Proof* The mapping  $\alpha_{\mathcal{V}} \circ \pi : \mathcal{U} \rightarrow X$  is also a  $\mathcal{U}$ -valuation since

$$(\alpha_{\mathcal{V}} \circ \pi)(s_{\mathcal{U}}(U)) = \alpha_{\mathcal{V}}(\pi(s_{\mathcal{U}}(U))) = \alpha_{\mathcal{V}}(s_{\mathcal{V}}(\pi(U))) = f(\alpha_{\mathcal{V}}(\pi(U))) = f((\alpha_{\mathcal{V}} \circ \pi)(U))$$

for all  $U \in \mathcal{U}^p$ , and  $(\alpha_{\mathcal{V}} \circ \pi)(\emptyset) = \alpha_{\mathcal{V}}(\pi(\emptyset)) = \alpha_{\mathcal{V}}(\emptyset) = \emptyset$ . By the uniqueness of the  $\mathcal{U}$ -valuation it therefore follows that  $\alpha_{\mathcal{U}} = \alpha_{\mathcal{V}} \circ \pi$ .  $\square$

**Proposition 12.7** If  $A \approx B$  then  $\alpha_{\mathcal{U}}(A) = \alpha_{\mathcal{V}}(B)$  for each  $A$ -enumerator  $\mathcal{U}$  and each  $B$ -enumerator  $\mathcal{V}$ . In particular,  $\alpha_{\mathcal{U}}(A) = \alpha_{\mathcal{U}'}(A)$  for all  $A$ -enumerators  $\mathcal{U}$  and  $\mathcal{U}'$ .

*Proof* By Theorem 12.3 there exists a unique homomorphism  $\pi : \mathcal{U} \rightarrow \mathcal{V}$  which maps  $\mathcal{U}$  bijectively onto  $\mathcal{V}$ , and  $\pi(A) = B$ . Also, by Proposition 12.6  $\alpha_{\mathcal{U}} = \alpha_{\mathcal{V}} \circ \pi$  and therefore  $\alpha_{\mathcal{U}}(A) = \alpha_{\mathcal{V}}(\pi(A)) = \alpha_{\mathcal{V}}(B)$ .  $\square$

**Theorem 12.5** For each  $A$ -enumerator  $\mathcal{U}$  we have  $\alpha_{\mathcal{U}}(A) = \omega(A)$ .

*Proof* If  $\mathcal{U}$  and  $\mathcal{U}'$  are  $A$ -enumerators then by Proposition 12.7  $\alpha_{\mathcal{U}}(A) = \alpha_{\mathcal{U}'}(A)$  and hence the element  $\alpha_{\mathcal{U}}(A)$  of  $X$  does not depend on which  $A$ -enumerator  $\mathcal{U}$  is used. Denote this element by  $\omega_*(A)$ . By the uniqueness of the valuation it is enough to show that the assignment  $A \mapsto \omega_*(A)$  is a valuation. Now if  $\mathcal{U}^{\emptyset}$  is the unique  $\emptyset$ -enumerator then  $\mathcal{U}^{\emptyset}(\emptyset) = x_0$  and so  $\omega_*(\emptyset) = x_0$ . Thus consider a finite set  $A$  and let  $a$  be an element with  $a \notin A$ ; put  $A' = A \cup \{a\}$ . By Lemma 12.6 there exists an  $A'$ -enumerator  $\mathcal{U}'$  with  $A \in \mathcal{U}'$  and then  $A' = s_{\mathcal{U}'}(A)$ , since  $a$  is the only element in  $A' \setminus A$ . Moreover  $\mathcal{U}'_A$  is an  $A$ -enumerator and  $\alpha_{\mathcal{U}'_A}$  is the restriction of  $\alpha_{\mathcal{U}'}$  to  $\mathcal{U}'_A$  and so

$$\omega_*(A') = \alpha_{\mathcal{U}'}(A') = \alpha_{\mathcal{U}'}(s_{\mathcal{U}'}(A)) = f(\alpha_{\mathcal{U}'}(A)) = f(\alpha_{\mathcal{U}'_A}(A)) = f(\omega_*(A)).$$



Hence  $\omega_*$  is a valuation and therefore  $\omega_* = \omega$ , i.e.,  $\alpha_{\mathcal{U}}(A) = \omega(A)$  for each finite set  $A$  and each  $A$ -enumerator  $\mathcal{U}$ .  $\square$

Consider the case with  $A \approx B$ . Let  $\mathcal{U}$  be an  $A$ -enumerator and  $\mathcal{V}$  a  $B$ -enumerator. By Proposition 12.7  $\alpha_{\mathcal{U}}(A) = \alpha_{\mathcal{V}}(B)$  and hence by Theorem 12.5

$$\omega(A) = \alpha_{\mathcal{U}}(A) = \alpha_{\mathcal{V}}(B) = \omega(B) .$$

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